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ON THE THEORY OF SEMI-IMPLICIT PROJECTION METHODS FOR VISCOUS INCOMPRESSIBLE FLOW AND ITS IMPLEMENTATION VIA A FINITE ELEMENT METHOD THAT ALSO INTRODUCES A NEARLY CONSISTENT MASS MATRIX. PART 1: THEORY

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SUMMARY

Ever since the time of Chorin's classic 1968 paper on projection methods, there have been lingering and poorly understood issues related to the best—or even proper or appropriate—boundary conditions (BCs) that should be (or could be) applied to the 'intermediate' velocity when the viscous terms in the incompressible Navier–Stokes equations are treated with an implicit time integration method and a Poisson equation is solved as part of a 'time step'. These issues also pervade all related methods that uncouple the equations by 'splitting' the pressure computation from that of the velocity—at least in the presence of solid boundaries and (again) when implicit treatment of the viscous terms is employed. This paper is intended to clarify these issues by showing which intermediate BCs are 'best' and why some that are not work well anyway. In particular we show that *all* intermediate BCs *must* cause problems related to the regularity of the solution near boundaries, but that a near-miraculous recovery occurs such that accurate results are nevertheless achieved beyond the *spurious* boundary layer *introduced* by such methods. The mechanism for this 'miracle' is related to the existence of a higher-order equation that is actually satisfied by the pressure. All that is required then for projection (splitting, fractional step, etc.) methods to work well is that the spurious boundary layer be thin—as has been largely observed in practice.

KEY WORDS Incompressible flow Navier–Stokes equations Projection methods Splitting methods Fractional step methods

1. INTRODUCTION

In this first part of a two-part paper we present the theory behind projection methods—a theory that also applies to numerous aliases: splitting methods, fractional step methods, pressure correction methods, to name a few. After defining the goal of these *approximations*, they are carefully derived in a fully continuous setting, in which the so-called optimal boundary conditions (BCs) are also derived. Next, these optimal BCs are waived and a family of simpler but less defensible schemes is presented. After a discussion on wall vorticity production, the simple schemes are extended to the case wherein the flow leaves the computational domain. The simpler schemes are then justified *a posteriori* by inverting—in principle—the sequence of steps. After presenting the semi-discrete analogues of the simpler projection methods wherein time is discretized, the final explanation of the success of these methods is unveiled by revealing the biharmonic equation for the pressure and the concomitant biharmonic miracle.

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2. THE CONTINUUM EQUATIONS

2.1. The conventional Navier–Stokes equations (the goal)

The equations of interest are the 2D (and 3D, but herein we focus on the former) time-dependent incompressible Navier–Stokes (NS) equations for the velocity ($\mathbf{u} = (u\ v)^T$) and kinematic pressure (P, pressure divided by density) in a bounded domain (Ω):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

where $\mathbf{f}$ is a given 'body force'. These equations are to be solved subject to the typical boundary conditions (BCs) of: specified velocity (Dirichlet) on Γ_1, i.e.

$$\mathbf{u} = \mathbf{w} \quad \text{on } \Gamma_1, \tag{1c}$$

and natural 'pseudo-traction' conditions (Neumann) on Γ_2, i.e.

$$-P + \nu\frac{\partial u_n}{\partial n} = F_n \quad \text{and} \quad \nu\frac{\partial u_\tau}{\partial n} = F_\tau \quad \text{on } \Gamma_2, \tag{1d}$$

where $\Gamma_1 \oplus \Gamma_2 = \partial\Omega$ (the boundary of Ω), n represents the outward normal direction ($u_n \equiv \mathbf{u}\cdot\mathbf{n}$), τ represents the corresponding tangential direction ($u_\tau \equiv \mathbf{u}\cdot\tau$) and F_n and F_τ are the normal and tangential components of the specified boundary 'traction' ($F_n = F_\tau = 0$ is commonly used when Γ_2 represents an 'outflow' boundary); and initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{1e}$$

where it is required that

$$\mathbf{n}\cdot\mathbf{u}_0 = \mathbf{n}\cdot\mathbf{w}(\mathbf{x}, 0) \quad \text{on } \Gamma_1 \tag{1f}$$

and

$$\nabla\cdot\mathbf{u}_0 = 0 \quad \text{in } \Omega \tag{1g}$$

in order that a solution exist. (The NS equations are ill-posed if either (1f) or (1g) is violated; see Gresho and Sani,[1] hereafter referred to as GS, for details.) Finally, if $\Gamma_2 = \varnothing$ (the null set) or, more generally, if $\mathbf{u}\cdot\mathbf{n}$ is specified on *all* of $\partial\Omega$, another solvability constraint (global mass conservation) enters:

$$\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{w}(\mathbf{x}, t) = 0. \tag{1h}$$

Equation (1) implies the following pressure Poisson equation (PPE),

$$\nabla^2 P = \nabla\cdot(\mathbf{f} - \mathbf{u}\cdot\nabla\mathbf{u}) \quad \text{in } \Omega, \tag{2a}$$

with concomitant BCs

$$\frac{\partial P}{\partial n} = \mathbf{n}\cdot\left(\nu\nabla^2\mathbf{u} + \mathbf{f} - \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\cdot\nabla\mathbf{u}\right) \quad \text{on } \Gamma_1 \tag{2b}$$

and

$$P = \nu\frac{\partial u_n}{\partial n} - F_n \quad \text{on } \Gamma_2. \tag{2c}$$

If all of the above solvability conditions are respected—see GS for details—the system given by (1) delivers the same solution $(\mathbf{u}, P)$ as that given by (1) with (1b) omitted and replaced by (2).

Remarks

1. BC (2b), whose relevance was re-established in GS, plays a major role in the theory of projection methods, in which it is a sought-after-but-never-quite-attained goal. This BC in fact 'causes' the normal component of the momentum equations to satisfy the first-order compatibility condition

$$\lim_{x\to\Gamma}\frac{\partial[\mathbf{n}\cdot\mathbf{u}(\mathbf{x},t)]}{\partial t}\bigg|_{t=0}=\lim_{t\to 0}\frac{\partial(\mathbf{n}\cdot\mathbf{u})}{\partial t}\bigg|_{\Gamma_1}=\mathbf{n}\cdot\dot{\mathbf{w}}_0,$$

   and is a consequence of the requirement that $\nabla\cdot\mathbf{u}=0$ on Γ—see Remark 2. In contrast, the tangential component(s) of the momentum equation usually do *not* satisfy the first-order compatibility condition(s) see also Remark 4.
2. In the weak from of (1) and (2) that is associated with (leads to) the finite element method, BCs (1d) and (2c) are enforced *weakly* in such a way that the divergence-free constraint (1b) is enforced *strongly*; i.e. $\nabla\cdot\mathbf{u}=0$ in Ω *and on* $\partial\Omega$. In fact, we may (and do) strengthen (1b) to read

$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \bar{\Omega}, \tag{1b}$$

   where $\bar{\Omega}\equiv\Omega+\partial\Omega$ (see GS), a result that will be used repeatedly in the sequel.
3. Another consequence of mass conservation is that $\int_{\partial\Omega}\mathbf{n}\cdot\mathbf{u}=0$ is *always* satisfied—via the constraint on the data given by (1h) if $\Gamma_2=\varnothing$, and as a property of the solution otherwise.
4. For $t>0$ the solution of (1) and (2) in fact satisfies the *overdetermined Neumann problem* for the pressure; i.e. the tangential component(s) of (1a) are then also satisfied on Γ_1 even though the pressure Poisson equation is solved by applying only the *normal* component; via (2a) and (2b).

2.2. *The Navier–Stokes equations viewed as projections*

Because we will be using projection methods to *approximate* the solution of the NS equations, it may first be fruitful to re-interpret (or *attempt* to re-interpret) them as projections. To this end we first rewrite the NS equations as

$$\frac{\partial\mathbf{u}}{\partial t}+\nabla P=\mathbf{S}(\mathbf{u}) \quad \text{in } \Omega, \qquad \nabla\cdot\mathbf{u}=0 \quad \text{in } \bar{\Omega}, \tag{3}$$

where $\mathbf{S}(\mathbf{u})\equiv\nu\nabla^2\mathbf{u}+\mathbf{f}-\mathbf{u}\cdot\nabla\mathbf{u}$ is generally neither divergence-free nor curl-free and, following Chorin,[2,3] invoke the following equivalent interpretation: given $\mathbf{u}$, the vector $\mathbf{S}(\mathbf{u})$ is 'known' and can be projected onto both the subspace of divergence-free vectors $(\partial\mathbf{u}/\partial t)$ and the subspace of curl-free vectors (∇P), a process that can be stated mathematically (and formally) as

$$\frac{\partial\mathbf{u}}{\partial t}=\wp\mathbf{S}(\mathbf{u}) \quad \text{and} \quad \nabla P=Q\mathbf{S}(\mathbf{u}), \tag{4a, b}$$

where $\wp$ and $Q(\equiv I-\wp)$ are projection operators: $\wp^2=\wp$, $Q^2=Q$ and $\wp Q=Q\wp=0$; $\wp$ projects any vector onto the null space of div—$\nabla\cdot\wp=0$—and Q projects any vector onto the null space of curl—$\nabla\times Q=0$. Thus the acceleration is rendered divergence-free, as required by the time derivative of (3b). In fact, comparing (3) with (4) yields the explicit form of these operators—at least (again) formally:

$$\wp = I - \nabla(\nabla^2)^{-1}\nabla\cdot = I - \mathrm{grad}(\nabla^2)^{-1}\,\mathrm{div}, \qquad Q = I - \wp = \mathrm{grad}(\nabla^2)^{-1}\,\mathrm{div}.$$

We note too that $\wp$ and Q also contain all appropriate boundary conditions; i.e. they are operators with the BCs 'built-in'. A further discussion of these $\wp$s and Qs is presented in the Appendix, the discrete portion of which will be defined in Part 2.

Consider now the following *approximate* representation, in which the pressure (gradient) is 'guessed at'—call it $\nabla\tilde{P}$—rather than coming from (2) or (3) or (4):

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t} = \mathbf{S}(\tilde{\mathbf{u}}) - \nabla\tilde{P}, \tag{5a}$$

in which the new pseudo-velocity $\tilde{\mathbf{u}}$ is generally not divergence-free because $\tilde{P} \neq P$. Suppose too that $\tilde{\mathbf{u}}$ starts from the same IC as $\mathbf{u}$ and is (continuously) projected onto the divergence-free subspace via $\wp$, and that we call the divergence-free result $\mathbf{v}$; i.e.

$$\mathbf{v} = \wp\tilde{\mathbf{u}}. \tag{5b}$$

(Note that there is no 'feedback': $\mathbf{v}$ from (5b) does not affect $\tilde{\mathbf{u}}$ from (5a) during its evolution.) On the *assumption* that $\mathbf{v} = \mathbf{u}$, one could save time and money by solving (5) instead of (3). But since the assumption is generally *not* very good (except perhaps near $t=0$), we (ultimately) follow Chorin[2,3] and consider the *discrete* time approximate solutions of (4) and (5) in which, at the beginning of each (small) time step, we take $\tilde{\mathbf{u}} = \mathbf{u}$. This controls/limits the amount of wandering from the divergence-free subspace that is permitted by $\tilde{\mathbf{u}}$; i.e. we recall that $\mathbf{S}$ is not a divergence-free vector in general, nor is $\nabla\tilde{P}$, nor is $\mathbf{S} - \nabla\tilde{P}$, so that an extended time integration of (5a) would be 'dangerous' in that $\mathbf{v}$ from (5b) would generally (because $\tilde{P} \neq P$) diverge ever further from the time integral of (4a); i.e.

$$\mathbf{v} - \mathbf{u}_0 = \int_0^t \wp\{\mathbf{S}[\tilde{\mathbf{u}}(\mathbf{x},\tau)] - \nabla\tilde{P}\}\,\mathrm{d}\tau \neq \int_0^t \wp\mathbf{S}[\mathbf{u}(\mathbf{x},\tau)]\,\mathrm{d}\tau = \mathbf{u} - \mathbf{u}_0.$$

For example, the implicit (backward) Euler method on (4) gives

$$(\mathbf{u}_{n+1} - \mathbf{u}_n)/\Delta t = \wp\mathbf{S}(\mathbf{u}_{n+1}), \qquad \nabla P_{n+1} = Q\mathbf{S}(\mathbf{u}_{n+1}),$$

and the same scheme on (5) gives

$$(\tilde{\mathbf{u}}_{n+1} - \mathbf{u}_n)/\Delta t = \mathbf{S}(\tilde{\mathbf{u}}_{n+1}) - \nabla\tilde{P}_{n+1},$$

followed by

$$\mathbf{v}_{n+1} = \wp\tilde{\mathbf{u}}_{n+1}.$$

The 'realization' of the *true* backward Euler projection is, however, not simple; it involves the *simultaneous* solution for velocity and pressure (i.e. it can only be *realized* by applying the backward Euler method to the coupled system in (3)). On the other hand, the realization of (5) is simpler because it is *sequential*—given that we have solved for $\tilde{\mathbf{u}}_{n+1}$ as above, the projection step is *realized* via the following decomposition:

$$\tilde{\mathbf{u}}_{n+1} = \mathbf{v}_{n+1} + \nabla\varphi, \qquad \nabla\cdot\mathbf{v}_{n+1} = 0,$$

where φ is a Lagrange multiplier associated with the projection of $\tilde{\mathbf{u}}_{n+1}$, and $\mathbf{v}_{n+1}$ is taken as an approximation to the solution of (3). The key word is *sequential*—$\mathbf{u}$ *then* P rather than $\mathbf{u}$ *and* P from (3) or (4)—and this is a primary reason for considering (implicit or semi-implicit) projection methods. It is also worth emphasizing that $\mathbf{v}_{n+1} \neq \mathbf{u}_{n+1}$ and that $\mathbf{u}_{n+1}$ will always be closer to $\mathbf{u}(t_{n+1})$, the true NS velocity; i.e. the backward Euler projection method is an approximation of an approximation. But before getting further into the semi-discrete backward Euler projection method, we back up and consider a whole family of projection methods, first in the *space–time continuum*, then in the semi-discrete case of which the above is one member, and finally (in Part 2) in the fully discrete case via a new finite element method.

The interim bottom line is this: while the acceleration and pressure *can* indeed be legitimately computed via the sequential steps associated with a projection, the *velocity* and pressure cannot—they are intimately coupled in an incompressible flow. But the philosophy of projection methods, in the hope of finding a cost-effective approximation, is 'try anyway', and this we do in the remainder of this paper, in which—at the end—the illegitimacy of the uncoupled approach is finally rationalized and largely justified.

3. THE PROJECTION METHOD APPROXIMATION

The more one studies the literature on and deeply contemplates the theory behind 'projection' methods (or fractional step methods, or splitting methods, or pressure correction methods, or velocity correction methods, or . . .), whose goal is to reduce the cost of a simulation by uncoupling the velocity from the pressure, the more one is led to the following two questions: (1) How do they really come about? (2) Why and how do they work? That is, the velocity and pressure are not really *meant* to be uncoupled for viscous incompressible flow! (For example, in a recent paper which is complementary to our own, Orszag *et al.*[4] state: 'While these steps have been well known to practitioners of computational fluid dynamics, it is still mysterious how and why they work.' For a recent exposé of our own confusion on this subject, as well as a valid discussion of some of the issues, see the BC discussion in Gresho and Chan.[5]) Herein we attempt to answer these questions, initially in the context of a family of methods that are continuous in both space and time. These are followed by semi-discrete methods, and finally, in the second part of the paper, by fully discrete realizations in the form of algorithms for the computer—some of which we demonstrate.

3.1. Derivation of (continuous) projection methods

3.1.1. Optimal schemes. We begin by restating the concept of a projection method, first in words and then in mathematics, noting that in our definition projection methods must *always* be viewed as techniques for obtaining *approximate* solutions of the NS equations. First, it is clear that if the proper (correct) pressure gradient were known as a function of space and time, then the NS equations would 'merely' represent a coupled system of vector 'heat equations' with the continuity equation being completely superfluous. With this as a general goal, projection method approximations generally proceed as follows:

(0) Given a divergence-free velocity field that satisfies the appropriate BCs, say at $t = 0$ for convenience, perform the following steps.
(1) Guess—i.e. approximate in some way—the concomitant pressure gradient, both at $t = 0$ and for $t > 0$.
(2) Solve the momentum equations *alone* up to 'projection time', $t = T$, which time could either be set *a priori* (by clever or otherwise methods) or could be defined as that time at which an appropriate norm of the divergence of the resulting 'intermediate velocity' reaches some predetermined maximum allowable value. (Its selection is clearly an important part of the approximation.)
(3) Perform the projection of the intermediate velocity onto the appropriate subspace of divergence-free vector fields. Call (perhaps brazenly) the result the desired (physical) velocity. This completes one projection cycle; reset time to $t = 0$ and go to Step (1).

Remark. 'Resetting the clock' ($t = 0$) is merely a convenience in the presentation. It of course does not imply resetting time-dependent body forces and/or BCs back to those at the beginning of the simulation. It merely means to go to the next projection cycle. In fact, all of the discussion to follow is actually directed toward the case $t > 0$, for which the NS solution satisfies the overdetermined Neumann problem—the vector momentum equation is satisfied in Ω and on Γ, a situation *not* shared by projection methods.

The realization of the above steps involves a sequence of 'details', some of which are rather important and are related to BCs—both on the intermediate velocity and on the physical velocity. Another important detail is that of guessing the pressure field. Needless to say, the most 'successful' projection methods (those whose projected velocity is close to the NS velocity for T 'large') will incorporate a good guess for the pressure and will apply good BCs.

We shall derive—and later analyse—one member of a family of projection methods, and at the end present two others in summary form. The method we shall derive in detail is the second in the family; we called it 'Projection 2' in our earlier publications on this subject[5,6] and will continue to do so herein. (An *explicit* version of Projection 2—via lumped mass finite elements—was introduced much earlier[7] but apparently not retained; i.e. in subsequent publications[8,9] these authors employed a technique closer to that of Chorin—except that they remained 'explicit' whereas Chorin and we use semi-implicit time integration—which we call Projection 1.)

Its mathematical description is as follows, wherein it is important to state that, until further notice, we restrict our attention to the simpler, fully Dirichlet BC—i.e. Γ_2 is $\varnothing$.

(*a*) *Intermediate velocity.* Given the same $\mathbf{u}_0$ as in (1), as well as the concomitant pressure field $P_0(\mathbf{x})$ *and* the rate of change of this pressure field on the boundary, $\dot{P}_0(\mathbf{x})$ for $\mathbf{x} \in \Gamma$, solve for the intermediate velocity $\tilde{\mathbf{u}}(\mathbf{x}, t)$ from

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \nabla P_0 = \nu \nabla^2 \tilde{\mathbf{u}} + \tilde{\mathbf{f}} \quad \text{in } \Omega, \tag{6a}$$

$$\tilde{\mathbf{u}} = \mathbf{w}(t) + t\nabla(\beta_1 P_0 + \beta_2 t \dot{P}_0) \quad \text{on } \Gamma, \quad \text{for } 0 < t \leq T, \tag{6b}$$

with $\tilde{\mathbf{u}} = \mathbf{u}_0$ at $t = 0$, where the dimensionless scalars β_1 and β_2 are to be determined during our search for 'optimal' BCs for $\tilde{\mathbf{u}}$. (Note that if either β_1 or β_2 is non-zero, the intermediate velocity will both penetrate Γ and slip along it.) Here $\tilde{\mathbf{f}} \equiv \mathbf{f}(\tilde{\mathbf{u}})$ is a generic forcing term, which includes the advection term. (We suppress $\mathbf{u} \cdot \nabla \mathbf{u}$ because all of the 'projection theory' to follow is actually primarily applicable to the Stokes equations; i.e. all of the significant aspects/difficulties are caused by the viscous terms (Laplacian).)

(*b*) *Projection.* With $\tilde{\mathbf{u}}(\mathbf{x}, T)$ available, perform the projection

$$\mathbf{v}(\mathbf{x}, T) = \wp \tilde{\mathbf{u}}(\mathbf{x}, T)$$

as follows: solve for $\mathbf{v}$ and φ from

$$\tilde{\mathbf{u}} = \mathbf{v} + \nabla \varphi \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \bar{\Omega}, \tag{7a, b}$$

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{w}(T) \cdot \mathbf{n} \quad \text{on } \Gamma, \tag{7c}$$

where we shall soon explain the selection of this boundary condition (Remark 2). This is equivalent to and is realized by the following two-step procedure.

(i) Solve for φ from

$$\nabla^2 \varphi = \nabla \cdot \tilde{\mathbf{u}} \quad \text{in } \Omega, \tag{8a}$$

$$\frac{\partial\varphi}{\partial n}=\mathbf{n}\cdot(\tilde{\mathbf{u}}-\mathbf{v}) = T\frac{\partial}{\partial n}(\beta_1 P_0+\beta_2 T\dot{P}_0) \quad \text{on } \Gamma. \tag{8b}$$

(ii) Compute

$$\mathbf{v} = \tilde{\mathbf{u}}-\nabla\varphi \quad \text{in } \bar{\Omega}. \tag{9}$$

(*c*) *Pressure update.* Accepting (or *defining*) $\mathbf{v}$ as the 'physical' velocity at time $t = T$, determine $P(T)$ and $\dot{P}(T)$—details to follow later—and reset the clock: $t = 0$, $\mathbf{v}(T)\rightarrow\mathbf{u}_0$ in Ω, $\mathbf{w}(T)\rightarrow\mathbf{u}_0$ on Γ, $P(T)\rightarrow P_0$ and $\dot{P}(T)\rightarrow\dot{P}_0$. One cycle of the projection method is thus complete—in principle. Before completing the derivation, we make the following.

Remarks

1. The BCs chosen for $\tilde{\mathbf{u}}$ are somewhat arbitrary because $\tilde{\mathbf{u}}$ is not quite a physical entity. Optimal BCs do exist, however, in some sense, as we shall see. Also, it is probably not fruitful to consider any higher-order terms in the BC for $\tilde{\mathbf{u}}$, since the proper ones would be 'impossible' to evaluate.
2. Since (7a, b) is rendered well-posed by specifying *either* normal *or* tangential components of $\mathbf{v}$ on Γ (but not both—in general), it would *seem* to follow that the BC chosen for $\mathbf{v}$, (7c), is not the only one that can be used; e.g. presumably one could replace (7c) by $\boldsymbol{\tau}\cdot\mathbf{v} = \boldsymbol{\tau}\cdot\mathbf{w}(T)$ on Γ. That this is presumption is false can be ascertained by examining its consequences in the light of the solvability requirement given by (1f); i.e. the computed solution $\mathbf{v}(x, T)$ would not satisfy $\mathbf{n}\cdot\mathbf{v}(x, T)=\mathbf{n}\cdot\mathbf{w}(x, T)$ on Γ at the end of the projection step—it would violate $\nabla\cdot\mathbf{v} = 0$ on Γ via $\mathbf{v}(T)\rightarrow\mathbf{u}_0$ on Γ and thus be ill-posed, and the next projection cycle could not then be performed. (It is perhaps worth emphasizing that setting $\mathbf{v} = \mathbf{w}$ on Γ in the *projection* step is an overspecified problem that generally has no solution; i.e. ϕ will not generally satisfy the overdetermined Neumann problem. See also Orszag and Israeli[10] and GS.) The confusion related to this very issue has in fact been part of the mystique of projection methods.
3. The BC for φ, (8b), follows from (7a, b); i.e. $\nabla\cdot\mathbf{v} = 0$ on Γ implies that the normal component of (7a) applies on Γ. In fact, the first of (8b) will *always* hold on Γ, regardless of the choice of intermediate velocity BC and regardless of whether or not $\Gamma_2=\varnothing$; again this is a consequence of the fact that $\nabla\cdot\mathbf{v} = 0$ on Γ as well as in Ω; see GS.
4. The solvability condition implied by the Neumann problem (8),

$$\int_\Gamma \mathbf{n}\cdot\tilde{\mathbf{u}} = \int_\Gamma T\frac{\partial}{\partial n}(\beta_1 P_0+\beta_2 T\dot{P}_0),$$

   is automatically satisfied if (1h) is satisfied.
5. Since $\nabla\times\nabla\varphi\equiv 0$, it seems clear at this point that the vorticity 'contained by' the intermediate velocity, $\nabla\times\tilde{\mathbf{u}}$, is unchanged by the projection (hopefully then, this vorticity is a good approximation to the true vorticity $\nabla\times\mathbf{u}$); i.e. the projection is—for the most part at least, before the (necessary) introduction of vortex sheets—a *potential* flow adjustment.
6. A seemingly deleterious but unavoidable by-product of projection methods is a spurious slip velocity. Whereas the true NS velocity will satisfy $\mathbf{u} = \mathbf{w}$ on Γ (which also causes the pressure to *satisfy* the overdetermined Neumann problem—except at the beginning of a simulation; see Heywood,[11] Heywood and Rannacher[12], and GS), the velocity from the projection method *approximation*, by construction, *must* 'slip' along Γ:

$$\boldsymbol{\tau}\cdot\mathbf{v} = \boldsymbol{\tau}\cdot(\tilde{\mathbf{u}}-\nabla\varphi)\neq\boldsymbol{\tau}\cdot\mathbf{w}$$

in general, so that another goal of projection methods is to minimize the slip velocity,

$$s \equiv \tau \cdot (\mathbf{v} - \mathbf{w}) \quad \text{on } \Gamma, \tag{10a}$$

so (in part) that 'wall vorticity production' will be as smooth and accurate as possible. Here τ is the unit tangent vector in a 2D domain; in 3D this can be written as

$$\mathbf{n} \times (\mathbf{v} \times \mathbf{n}) \neq \mathbf{n} \times (\mathbf{w} \times \mathbf{n}) \quad \text{and} \quad \mathbf{s} \equiv \mathbf{n} \times [(\mathbf{v} - \mathbf{w}) \times \mathbf{n}] \quad \text{on } \Gamma. \tag{10b}$$

Part of the vorticity production is in fact realized (albeit discontinuously, in the form of a vortex sheet) *after* the projection step, by setting $\mathbf{u}_0 = \mathbf{w}(T)$ on Γ rather than $\mathbf{u}_0 = \mathbf{v}(T)$ to begin the next cycle. We will have more to say on this later.

7. If $\mathbf{n} \cdot \mathbf{w} = 0$, the projection is an *orthogonal* projection and satisfies a Pythagorean theorem:[13]

$$\int_\Omega \mathbf{v} \cdot \nabla\varphi = 0 \quad \text{and} \quad \|\tilde{\mathbf{u}}\|^2 = \|\mathbf{v}\|^2 + \|\nabla\varphi\|^2,$$

where

$$\|\mathbf{u}\|^2 \equiv \int_\Omega \mathbf{u} \cdot \mathbf{u}.$$

(*Proof*:

$$\begin{aligned}\int_\Omega \varphi \nabla \cdot \mathbf{v} &= 0 \\ &= \int_\Omega \nabla \cdot (\varphi \mathbf{v}) - \int_\Omega \mathbf{v} \cdot \nabla\varphi \\ &= \int_\Gamma \varphi \mathbf{n} \cdot \mathbf{v} - \int_\Omega \mathbf{v} \cdot \nabla\varphi \\ &= \int_\Gamma \varphi \mathbf{n} \cdot \mathbf{w} - \int_\Omega \mathbf{v} \cdot \nabla\varphi \\ &= -\int_\Omega \mathbf{v} \cdot \nabla\varphi.\end{aligned}$$

Then

$$\begin{aligned}\|\tilde{\mathbf{u}}\|^2 = \int_\Omega (\mathbf{v} + \nabla\varphi) \cdot (\mathbf{v} + \nabla\varphi) &= \int_\Omega \mathbf{v} \cdot \mathbf{v} + 2\int_\Omega \mathbf{v} \cdot \nabla\varphi + \int_\Omega \nabla\varphi \cdot \nabla\varphi \\ &= \|\mathbf{v}\|^2 + \|\nabla\varphi\|^2.)\end{aligned}$$

Note also the interesting and perhaps counter-intuitive result that

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \nabla P = 0$$

when $\mathbf{n} \cdot \mathbf{w} = 0$ on Γ, where $\mathbf{u}$ and P are the NS solution; the acceleration is, 'on the average', orthogonal to the pressure gradient when the velocity is parallel to the boundaries (i.e. for $x \to \Gamma$); see also Chorin and Marsden.[14]

The next step in the derivation is to compare $\tilde{\mathbf{u}}(T)$ with the NS velocity $\mathbf{u}(T)$ at the same time. This we begin by applying simple Taylor series analyses as follows:

$$\mathbf{u}(t) = \mathbf{u}_0 + t\dot{\mathbf{u}}_0 + \frac{t^2}{2}\ddot{\mathbf{u}}_0 + O(t^3),$$

$$\tilde{\mathbf{u}}(t) = \tilde{\mathbf{u}}_0 + t\dot{\tilde{\mathbf{u}}}_0 + \frac{t^2}{2}\ddot{\tilde{\mathbf{u}}}_0 + O(t^3), \tag{11}$$

where $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$, and we invoke the PDEs to get—*assuming sufficient smoothness*—

$$\dot{\mathbf{u}}_0 = \dot{\tilde{\mathbf{u}}}_0 = \nu\nabla^2\mathbf{u}_0 + \mathbf{f}_0 - \nabla P_0,$$

$$\ddot{\mathbf{u}}_0 = \frac{\partial}{\partial t}[\nu\nabla^2\mathbf{u} + \mathbf{f} - \nabla P]_{t=0} = \nu\nabla^2\dot{\mathbf{u}}_0 + \dot{\mathbf{f}}_0 - \nabla\dot{P}_0,$$

$$\ddot{\tilde{\mathbf{u}}}_0 = \frac{\partial}{\partial t}[\nu\nabla^2\tilde{\mathbf{u}} + \tilde{\mathbf{f}} - \nabla P_0]_{t=0} = \nu\nabla^2\dot{\mathbf{u}}_0 + \dot{\tilde{\mathbf{f}}}_0, \quad \text{where } \dot{\mathbf{f}}(\mathbf{u}) \equiv \frac{\partial\mathbf{f}}{\partial\mathbf{u}}\cdot\dot{\mathbf{u}},$$

to give, after verifying that $\dot{\tilde{\mathbf{f}}}_0 = \dot{\mathbf{f}}_0$,

$$\tilde{\mathbf{u}}(t) - \mathbf{u}(t) = \frac{t^2}{2}\nabla\dot{P}_0 + O(t^3), \tag{12}$$

a result that is generally true in Ω but may break down for $x \to \Gamma$ because of a BC incompatibility, which we examine next.

Inserting (12) into (8) at $t = T$ gives

$$\nabla^2\varphi = \nabla^2\left(\frac{T^2}{2}\dot{P}_0 + O(T^3)\right) \quad \text{in } \Omega, \tag{13a}$$

$$\begin{aligned}\frac{\partial\varphi}{\partial n} &= \mathbf{n}\cdot[\tilde{\mathbf{u}}(T) - \mathbf{w}(T)] \\ &= T\frac{\partial}{\partial n}(\beta_1 P_0 + \beta_2 T\dot{P}_0) \\ &= \frac{\partial}{\partial n}\left(\frac{T^2}{2}\dot{P}_0 + O(T^3)\right) \quad \text{on } \Gamma;\end{aligned} \tag{13b}$$

with the last result a consequence of *assuming* that the normal component of (12) holds on Γ, which we now examine more closely. These *two* Neumann BCs on φ amount to a *compatibility condition on the boundary*; i.e. they imply that

$$\frac{\partial}{\partial n}[\beta_1 TP_0 + (\beta_2 - \tfrac{1}{2})T^2\dot{P}_0] = O(T^3)$$

should be satisfied, and this leads to the selection $\beta_1 = 0$, $\beta_2 = \frac{1}{2}$ as the coefficients that yield the '*optimal*' *BCs on the intermediate velocity*; i.e. the BC on $\tilde{\mathbf{u}}$ is

$$\tilde{\mathbf{u}} = \mathbf{w} + \tfrac{1}{2}t^2\nabla\dot{P}_0. \tag{6c}$$

This selection in fact leads to the satisfaction of the first- and second-order compatibility conditions for (6); i.e. it yields continuous values of $\partial\tilde{\mathbf{u}}/\partial t$ and $\partial^2\tilde{\mathbf{u}}/\partial t^2$ for $t \to 0$ and $x \to \Gamma$. Higher-order compatibility conditions, however, are *not* satisfied, even with this optimal selection, e.g.

$$\lim_{t\to 0}\frac{\partial^3\tilde{\mathbf{u}}}{\partial t^3}\bigg|_\Gamma \neq \lim_{t\to\Gamma}\frac{\partial^3\tilde{\mathbf{u}}_0}{\partial t^3} \equiv \lim_{x\to\Gamma}\frac{\partial^2}{\partial t^2}[\nu\nabla^2\tilde{\mathbf{u}}+\tilde{\mathbf{f}}-\nabla P_0]_{t=0},$$

an observation that will be more relevant when we discuss simpler projection methods. Hence the intermediate velocity, even from this 'optimal' projection method, like that from *all other projection methods*, will always suffer some loss of regularity (relative to the true NS velocity) near Γ for $t\to 0$: in general, a boundary layer (BL) will exist—details to follow later—in which some of the derivatives of $\tilde{\mathbf{u}}$ experience large changes.

With this selection of the β_i it then follows that the solution of (13) is

$$\varphi = \frac{T^2}{2}\dot{P}_0 + O(T^3) \quad \text{in } \bar{\Omega}, \tag{14a}$$

and we see how the Lagrange multiplier of this optimal projection method is related to the pressure, an important result that we shall return to later when we present simpler and *cost-effective* projection methods. (The method we are currently deriving makes no such claim.)

Finally, we examine $\mathbf{v} = \tilde{\mathbf{u}}(T) - \nabla\varphi$ from (9), and easily see, using (12) and (14a), that

$$\mathbf{v} = \mathbf{u}(T) + O(T^3) \quad \text{in } \Omega; \tag{14b}$$

i.e. $\mathbf{v}$ appears to be a good approximation to the NS velocity—at least 'away from' Γ.

We now present an algorithmic statement of a fully continuous and 'optimal' Projection 2 method summarizing what has thus far been derived and filling in the missing steps.

Remark. Below and henceforth we denote, for simplicity, an entire algorithm by what might usually be considered an 'equation number'.

Optimal projection 2 (15)

(0) Given $\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0 = 0$, P_0 and $\dot{P}_0$,

(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t} - \nu\nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \nabla P_0 \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} + \frac{t^2}{2}\nabla\dot{P}_0 \quad \text{in } \Gamma, \quad \text{for } 0 < t \leq T.$$

(2) Solve for φ from

$$\nabla^2\varphi = \nabla\cdot\tilde{\mathbf{u}}(T) \quad \text{in } \Omega,$$

$$\frac{\partial\varphi}{\partial n} = \frac{T^2}{2}\frac{\partial\dot{P}_0}{\partial n} \quad \text{on } \Gamma.$$

(3) Compute

$$\mathbf{v} = \tilde{\mathbf{u}}(T) - \nabla\varphi \quad \text{in } \bar{\Omega}.$$

(4) Solve for $P(T)$ from

$$\nabla^2 P = \nabla\cdot\mathbf{f}(\mathbf{v}) \quad \text{in } \Omega,$$

$$\frac{\partial P}{\partial n} = \mathbf{n}\cdot[\nu\nabla^2\mathbf{v}+\mathbf{f}-\dot{\mathbf{w}}(T)] \quad \text{on } \Gamma.$$

(5) Solve for $\dot{P}(T)$ from

$$\nabla^2\dot{P} = \nabla\cdot\dot{\mathbf{f}}(\mathbf{v}) \quad \text{in } \Omega,$$

$$\frac{\partial \dot{P}}{\partial n} = \mathbf{n}\cdot[\nu\nabla^2\dot{\mathbf{v}}+\dot{\mathbf{f}}-\ddot{\mathbf{w}}(T)] \quad \text{on } \Gamma,$$

where $\dot{\mathbf{v}} \equiv \nu\nabla^2\mathbf{v}+\mathbf{f}(\mathbf{v})-\nabla P$ and $\dot{\mathbf{f}}(\mathbf{v}) \equiv (\partial\mathbf{f}/\partial\mathbf{v})\cdot\dot{\mathbf{v}}$.

(6) Report $\mathbf{v}$ and P, then set $t=0$, $\mathbf{u}_0=\mathbf{v}$ (*except* in the tangential direction(s) on Γ; see Remark 1 below), $P_0=P$, $\dot{P}_0=\dot{P}$ and go to Step (1).

Remarks

1. $\mathbf{u}_0$ is not set to $\mathbf{v}(T)$ on Γ to begin the next cycle, which displays slip, but rather to $\mathbf{w}(T)$—a necessary procedure that introduces into $\tilde{\mathbf{u}}$ a vortex sheet at Γ of strength $s=O(T^3)$, where s is the slip velocity discussed earlier; see (10). The loss of regularity associated with this process (i.e. with a jump in the tangential velocity at Γ) is discussed by Heywood and Rannacher[12]—albeit for the true NS equations.
2. Start-up (the beginning of a simulation, not the beginning of a projection cycle) is not simple and seems to exact more smoothness from the data than do the NS equations; and, in general, the computation of $\dot{P}$ is unattractive.
3. Three Poisson equations per cycle seems a lot unless the method would work well for fairly large T. In fact:
4. A replacement of the PPE for $P(T)$ by $P(T)\equiv P_0+T\dot{P}_0$ and of the Poisson equation for $\dot{P}(T)$ by $\dot{P}(T)\equiv 2\varphi/T^2$ would (though inexpensive) not be optimal in that both $\partial P/\partial n$ and $\partial\dot{P}/\partial n$ on Γ would be wrong: after m projection cycles they would be $\partial P(mT)/\partial n = \partial P_0/\partial n + mT\partial\dot{P}_0/\partial n$ and $\partial\dot{P}(mT)/\partial n = \partial\dot{P}_0/\partial n$ respectively, where here P_0 and $\dot{P}_0$ refer to the values at the *beginning* of the simulation.

By following similar and hopefully obvious steps, other members of this family of projection methods can be derived. So, before pushing on towards more useful schemes, we first present a lower-order and a higher-order member of this family, again in the form of algorithms.

Optimal projection 1 (16)

(0) Given $\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0=0$ and P_0,

(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0=\mathbf{u}_0$ at $t=0$, from

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t}-\nu\nabla^2\tilde{\mathbf{u}}=\tilde{\mathbf{f}} \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}}=\mathbf{w}+t\nabla P_0 \quad \text{on } \Gamma, \quad \text{for } 0<t\le T.$$

(2) Solve for φ from

$$\nabla^2\varphi=\nabla\cdot\tilde{\mathbf{u}}(T) \quad \text{in } \Omega,$$

$$\frac{\partial\varphi}{\partial n}=T\frac{\partial P_0}{\partial n} \quad \text{on } \Gamma.$$

(3), (4) As in Projection 2.
(5) As in step (6) of Projection 2 except omit $\dot{P}_0 = \dot{P}$.

Remarks

1. At the beginning of a simulation, P_0 is obtained by solving (2).
2. This is (close to) the continuous form of Chorin's original projection method[2,3] and the simplest one possible, since it makes a guess of *zero* for $\nabla\tilde{P}$.
3. $\tilde{\mathbf{u}}(T) = \mathbf{u}(T) + T\nabla P_0 + O(T^2)$.
4. $\mathbf{v} = \mathbf{u}(T) + O(T^2)$ and $\varphi = TP_0 + O(T^2) = TP(T) + O(T^2)$.
5. The slip velocity (vortex sheet strength) is $O(T^2)$, but $\mathbf{n}\cdot\mathbf{v} = \mathbf{n}\cdot\mathbf{w}$ on Γ.
6. Only the first-order compatibility condition is satisfied. At second order,

$$\lim_{x\to\Gamma}\frac{\partial^2\tilde{\mathbf{u}}_0}{\partial t^2} \neq \lim_{t\to 0}\frac{\partial^2\tilde{\mathbf{u}}}{\partial t^2}\bigg|_\Gamma,$$

so that this scheme is less smooth than Projection 2.
7. *If* the result $\varphi = TP(T) + O(T^2)$ is used to replace solving the PPE of step (4) by $P(T) \equiv \varphi/T$, which is then to be used as P_0 in the next cycle—*à la* Chorin[2] and Kim and Moin[15]—the scheme is *no longer optimal* in the following sense: the inhomogeneous BC used for the projection, $\partial\varphi/\partial n = T\partial P_0/\partial n$, will never change its value from one cycle to the next, with the result that the *initial* normal pressure gradient is enforced *throughout* the computation. The fact that these schemes 'worked' is attributed to the biharmonic miracle, which we shall define later.

The next, a higher-order scheme (and the last that we consider), is:

Optimal projection 3 (17)

(0) Given $\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0 = 0$, P_0, $\dot{P}_0$ and $\ddot{P}_0$,
(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t} - \nu\nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \nabla(P_0 + t\dot{P}_0) \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} + \frac{t^3}{6}\nabla\ddot{P}_0 \quad \text{on } \Gamma, \quad \text{for } 0 < t \leq T.$$

(2) Solve for φ from

$$\nabla^2\varphi = \nabla\cdot\tilde{\mathbf{u}}(T) \quad \text{in } \Omega,$$

$$\frac{\partial\varphi}{\partial n} = \frac{T^3}{6}\frac{\partial\ddot{P}_0}{\partial n} \quad \text{on } \Gamma.$$

(3)–(5) As in Projection 2.
(6) Compute $\ddot{P}$ from

$$\nabla^2\ddot{P} = \nabla\cdot\ddot{\mathbf{f}}(\mathbf{v}) \quad \text{in } \Omega,$$

$$\frac{\partial\ddot{P}}{\partial n} = \mathbf{n}\cdot[\nu\nabla^2\ddot{\mathbf{v}} + \ddot{\mathbf{f}}(\mathbf{v}) - \ddot{\mathbf{w}}] \quad \text{on } \Gamma.$$

(7) Same as step (6) of Projection (2) except add $\ddot{P}_0 = \ddot{P}(T)$.

Remarks

1. The optimal BC derivation required the addition of $\beta_3 t^3 \nabla \ddot{P}_0$ to the RHS of (6b) and the selection of $\beta_1 = \beta_2 = 0$, $\beta_3 = \frac{1}{6}$.
2. $\tilde{\mathbf{u}}(T) = \mathbf{u}(T) + (T^3/6)\nabla \ddot{P}_0 + O(T^4)$.
3. $\mathbf{v} = \mathbf{u}(T) + O(T^4)$ and $\varphi = (T^3/6)\ddot{P}_0 + O(T^4)$.
4. The slip velocity (vortex sheet strength) is $O(T^4)$, but $\mathbf{n}\cdot\mathbf{v} = \mathbf{n}\cdot\mathbf{w}$ on Γ.
5. The first *three* compatibility conditions are satisfied—and a smoother yet solution will be realized. The first compatibility condition violation is at fourth order:
$$\lim_{x\to\Gamma} \frac{\partial^4 \tilde{\mathbf{u}}_0}{\partial t^4} \neq \lim_{t\to 0} \frac{\partial^4 \tilde{\mathbf{u}}}{\partial t^4}\bigg|_\Gamma .$$
6. The additional smoothness implications at start-up and the addition of yet one more Poisson equation per cycle are clearly undesirable features.
7. We employed a scheme similar to this (but with different BCs on $\tilde{\mathbf{u}}$) some years ago,[16] which we called subcycling.
8. Again start-up is difficult: $\dot{P}_0$ and (especially) $\ddot{P}_0$ are not readily available and smoother (than for NS) data are required.

A general remark concerning all of these optimal projection schemes is that they require the solution of more Poisson equations than most people would care to do; one (or less!) per projection cycle is much more appealing, and is in fact what most effective schemes in practice have been based upon. Add to this the 'inconvenience' of needing to compute $\partial P/\partial t$ for Projection 2 and $\partial^2 P/\partial t^2$ for Projection 3, and the so-called 'optimal' methods look less optimal. In fact, the only reason we called them optimal is that they can yield smoother solutions near Γ by virtue of the selection of (optimal) BCs on the intermediate velocity such that a high-order boundary compatibility condition is satisfied.

3.1.2. Simpler schemes. The above line of reasoning, combined with the two facts

(i) many practitioners have already 'done it'
(ii) the computation of ∇P on Γ is not easy to do in a finite element code

and the *suspicion* that P should be obtainable from φ, leads one to ask the question: 'Suppose we waive/violate/ignore these compatibility conditions to generate simpler algorithms?' This we do below; in particular, each of the (simpler, *and* advocated) schemes presented next uses the prescribed *physical* velocity as the BC on the intermediate velocity, which necessarily (via $\beta_i = 0$ in (6b) and (8b)) leads to *homogeneous* Neumann BCs for the *single* Poisson equation per cycle, for φ, and—initially at least—to a host of new 'problems'. (We remark that, in contrast to Kim and Moin[15] we do not believe in, nor have we experienced, problems related to 'consistency' with these BCs—only 'regularity'.) The simplified projection techniques presented now are justified later.

Projection 1 (18)

(0) Given $\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0 = 0$,
(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from
$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu\nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} \quad \text{in } \Omega,$$
$$\tilde{\mathbf{u}} = \mathbf{w} \quad \text{on } \Gamma, \quad \text{for } 0 < t \leq T.$$

(2) Solve for φ from

$$\nabla^2\varphi = \nabla\cdot\tilde{\mathbf{u}}(T) \quad \text{in } \Omega,$$

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \Gamma.$$

(3) Compute $\mathbf{v} = \tilde{\mathbf{u}}(T) - \nabla\varphi$ in $\bar{\Omega}$.

(4) Report $\mathbf{v}$; then set $t = 0$, $\mathbf{u}_0 = \mathbf{v}$ (except on Γ, as discussed earlier) and go to step (1).

Remarks

1. No pressure calculation (no PPE) is even *required*; but if a pressure estimate is desired, it can be obtained from $P(T) = \varphi/T$ by *assuming* that the same relationship obtained for the optimal scheme still applies—an assumption we will validate later.
2. The BC implied for the *implied* PPE for this scheme is $\partial P/\partial n = 0$ on Γ, which is clearly a *bad* BC vis-à-vis (2b); the scheme thus appears to be both hare-brained and doomed. (For example, consider a steady boundary-driven Stokes flow with no body force, for which (2) becomes $\nabla^2 P = 0$ in Ω, $\partial P/\partial n = \mathbf{n}\cdot\nu\nabla^2\mathbf{u}$ on Γ. The actual use of $\partial P/\partial n = 0$ here is clearly wrong since it implies $P = 0$.) But this is *only* an appearance, as we will demonstrate.
3. The slip velocity (vortex sheet strength) is larger, namely $s = -T\partial P_0/\partial\tau + O(T^2)$, but $\mathbf{n}\cdot\mathbf{v} = \mathbf{n}\cdot\mathbf{w}$ (still) on Γ. And in fact, *all* of the 'wall vorticity' is introduced into the fluid in this (discontinuous) manner.
4. Not even the first-order compatibility condition is satisfied; in fact,

$$\lim_{x\to\Gamma}\frac{\partial\tilde{\mathbf{u}}_0}{\partial t} - \lim_{t\to 0}\frac{\partial\tilde{\mathbf{u}}}{\partial t}\bigg|_\Gamma = (\mathbf{f}_0 + \nu\nabla^2\mathbf{u}_0)_\Gamma - \dot{\mathbf{w}}_0 = \nabla P_0|_\Gamma.$$

5. As previously mentioned, some (e.g. Chorin[2] and Kim and Moin[15]) have endeavored to use the optimal BC on $\tilde{\mathbf{u}}$, giving $\partial\varphi/\partial n = T\partial P_0/\partial n$ in the above algorithm. But a little inductive reasoning reveals that this BC merely holds $\partial P/\partial n$ on Γ at its *initial* (beginning of simulation) value, so this BC too seems to be bad—at least in the normal direction. But it *does* do a better job in the tangential direction, giving $s = O(T^2)$. These facts have in fact been put to good use by Fortin *et al.*[17] and Zang and Hussaini,[18] who employed a mix of simple and optimal BCs for $\tilde{\mathbf{u}}$ in their Projection 1 schemes; namely $\mathbf{n}\cdot\tilde{\mathbf{u}} = \mathbf{n}\cdot\mathbf{w}$ and $\tau\cdot\tilde{\mathbf{u}} = \tau\cdot\mathbf{w} + T\partial P_0/\partial\tau$, the former of course leading to the bad BC on pressure, $\partial P/\partial n = 0$ on Γ, but the latter leading to a smaller slip velocity.

Projection 2 (19)

(0) Given $\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0 = 0$ and P_0,

(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t} - \nu\nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \nabla P_0 \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} \quad \text{on } \Gamma, \quad \text{for } 0 < t \le T.$$

(2), (3) Same as Projection 1 in (18); i.e. perform the projection.

(4) Compute $P(T) = P_0 + 2\varphi/T$ in Ω.

(5) Report $\mathbf{v}$ and P; then set $t = 0$, $\mathbf{u}_0 = \mathbf{v}$ (but only *in* Ω, as previously discussed), $P_0 = P(T)$ and go to step (1).

Remarks

1. At the beginning of a simulation, P_0 is obtained by solving (2).
2. The $P(T)$ computation is now based on the *assumption* (see (14a)) that $\varphi \simeq (T^2/2)\dot{P}_0$, another assumption that we will soon validate—and the simple approximation $\dot{P}_0 = [P(T) - P_0]/T$.
3. The BC implied for the *implied* PPE is $\partial P(t)/\partial n = \partial P_0/\partial n$, which implies, by induction, that $\partial P/\partial n$ on Γ is *held* at its *initial* value—another *seemingly* worthless approximation. But this Projection 2 scheme actually generates very good results, and this paradox too will be explained in due course.
4. The slip velocity (vortex sheet strength) is $s = -(T^2/2)\partial \dot{P}_0/\partial \tau + O(T^3)$, whose cancellation adds a small amount of *additional* vorticity to the fluid, the major portion now coming from ∇P_0 during the intermediate velocity phase—i.e. the dominant *and* smoothly inserted vorticity 'comes from' $\partial P_0/\partial \tau|_\Gamma$ and the no-slip BC ($\tilde{\mathbf{u}} = \mathbf{w}$ on Γ). Also, $\mathbf{n}\cdot\mathbf{v} = \mathbf{n}\cdot\mathbf{w}$ (still) on Γ.
5. Only the first-order compatibility condition is satisfied. At second order,

$$\lim_{x\to\Gamma} \frac{\partial^2 \tilde{\mathbf{u}}_0}{\partial t^2} - \lim_{t\to 0} \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2}\bigg|_\Gamma = \nabla \dot{P}_0|_\Gamma.$$

Projection 3 (20)

(0) Given $\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0 = 0$, P_0, and $\dot{P}_0$,

(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu\nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \nabla(P_0 + t\dot{P}_0) \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} \quad \text{on } \Gamma, \quad \text{for } 0 < t \le T.$$

(2), (3) Same as Projection 1 in (18); i.e. perform the projection.

(4) Compute $P(T) = P_0 + T\dot{P}_0 + 3\varphi/T$ in $\bar{\Omega}$.

(5) Compute $\dot{P}(T) = [P(T) - P_0]/T = \dot{P}_0 + 3\varphi/T^2$ in $\bar{\Omega}$.

(6) Report $\mathbf{v}$ and P; then set $t = 0$, $\mathbf{u}_0 = \mathbf{v}$ (in Ω), $P_0 = P(T)$, $\dot{P}_0 = \dot{P}(T)$ and go to step (1).

Remarks

1. At the beginning of a simulation, P_0 is obtained by solving (2) and $\dot{P}_0$ is obtained by taking one very small step (Δt_s) using forward Euler or Projection 1 and using $\dot{P}_0 \equiv (P_1 - P_0)/\Delta t_s$.
2. The pressure calculation is now based on the *assumption* (again, validated later) that $\varphi \simeq (T^3/6)\ddot{P}_0$, along with (the fact that) $P(T) = P_0 + T\dot{P}_0 + (T^2/2)\ddot{P}_0 + O(T^3)$.
3. The BC implied for the *implied* PPE is $\partial P(T)/\partial n = \partial P_0/\partial n + T\partial\dot{P}_0/\partial n$, which (finally) is reasonable.
4. The slip velocity is $s = -(T^3/6)\partial\ddot{P}_0/\partial\tau + O(T^4)$. Also, $\mathbf{n}\cdot\mathbf{v} = \mathbf{n}\cdot\mathbf{w}$ (still) on Γ.
5. The third-order compatibility condition has been lost; in fact,

$$\lim_{x\to\Gamma} \frac{\partial^3 \tilde{\mathbf{u}}_0}{\partial t^3} - \lim_{t\to 0} \frac{\partial^3 \tilde{\mathbf{u}}}{\partial t^3}\bigg|_\Gamma = \nabla \ddot{P}_0|_\Gamma.$$

6. The 'crude' estimation of $\dot{P}$ in step (5)—in lieu of solving another Poisson equation—is judged to be cost-effective.
7. This scheme has not yet been tested in practice, but probably should be.

Before returning to the ostensibly worthless pressure BCs of Projections 1 and 2, we open Pandora's box even further by returning briefly to the thus-far ignored question: 'How is φ ***really*** related to P in the absence of the satisfaction of the maximum number of BC compatibility conditions?' Recall that we merely *assumed* that $\varphi \simeq TP$, $\varphi \simeq (T^2/2)\dot{P}$ and $\varphi \simeq (T^3/6)\ddot{P}$ to *define* the three simpler schemes. Consider Projection 2, the others (again) following by analogy. Whereas we determined that

$$\varphi = \frac{T^2}{2}\dot{P}_0 + O(T^3)$$

from the problem,

$$\nabla^2\varphi = \nabla^2\left(\frac{T^2}{2}\dot{P}_0 + O(T^3)\right) \quad \text{in } \Omega,$$

$$\frac{\partial\varphi}{\partial n} = \frac{\partial}{\partial n}\left(\frac{T^2}{2}\dot{P}_0 + O(T^3)\right) \quad \text{on } \Gamma,$$

when we used $\beta_2 = \frac{1}{2}$ in the optimal Neumann BC, now we have $\beta_2 = 0$ and thus $\partial\varphi/\partial n = 0$ on Γ, and we are led to seek a solution of the following form:

$$\varphi = \frac{T^2}{2}\dot{P}_0 + O(T^3) + F,$$

where F is a harmonic function; i.e. F must satisfy

$$\nabla^2 F = 0 \quad \text{in } \Omega, \qquad \frac{\partial F}{\partial n} = -\frac{T^2}{2}\frac{\partial \dot{P}_0}{\partial n} \quad \text{on } \Gamma.$$

Unfortunately, however, this new and generally unknown function is seen to be of size $O(T^2)$, which is 'too large'—i.e. it pollutes the presumed φ–P relationship and seems to *further undermine* the simpler projection methods.

The resolution of the above dilemma, and the paradoxes that preceded it, is contained in the results of an *analysis* of these projection methods—presented below—that ultimately leads to what we will call the biharmonic miracle. Rather than satisfying the PPE (2) of the true NS equations, the pressure from a projection method actually satisfies a higher-order equation—a sort of biharmonic equation—that contains the following 'miracle': it permits the pressure both to satisfy the 'bad BCs' mentioned above *and* to recover to the *proper* NS solution (through a boundary layer, or penetration depth) a short distance away from Γ. (It will be a *true* biharmonic equation when time is discretized.) The associated boundary layer thickness over which the miracle occurs is $\delta \equiv \sqrt{(\nu T)}$ and is of course 'sufficiently small' only when $\delta \ll l$, where l is any relevant physical length scale.

So one restriction on the projection cycle time T is $T \ll l^2/\nu$.

After two brief diversions, one on vorticity and the other regarding more general BCs, we will return to and prove these allegations.

3.1.3. Vorticity production. The issue of 'vorticity production at no-slip walls' is an important one, and it is therefore relevant to address it when using any *approximate* solution method. In this subsection we show how the post-projection cancellation of the slip velocity (thus far tentatively labelled 'spurious') is related to vorticity 'production/injection' at the boundary. In 2D (for simplicity) let $\omega \equiv \partial u_\tau/\partial n - \partial u_n/\partial \tau$ be the vorticity on Γ; then $Q \equiv -\nu\partial\omega/\partial n$ is the vorticity flux into Ω at Γ. Using $\nabla\cdot\mathbf{u} = 0 = \partial u_n/\partial n + \partial u_\tau/\partial\tau$, it follows that $Q = -\nu\nabla^2 u_\tau$, so that the total (NS) vorticity flowing into Ω at a point on Γ during time T is, from (1a), letting $g \equiv \tau\cdot[\mathbf{f} - \mathbf{u}\cdot\nabla\mathbf{u}]$ and focusing on the tangential pressure gradient,

$$\int_0^T Q\mathrm{d}t = \int_0^T (g - \partial P/\partial\tau)\mathrm{d}t - [w_\tau(T) - w_\tau(0)]$$
$$= \int_0^T g\mathrm{d}t - [w_\tau(T) - w_\tau(0)] - \int_0^T \left(\frac{\partial P_0}{\partial\tau} + t\frac{\partial \dot{P}_0}{\partial\tau} + \frac{t^2}{2}\frac{\partial \ddot{P}_0}{\partial\tau} + O(T^3)\right)\mathrm{d}t$$
$$= \int_0^T g\mathrm{d}t - [w_\tau(T) - w_\tau(0)] - \left(T\frac{\partial P_0}{\partial\tau} + \frac{T^2}{2}\frac{\partial \dot{P}_0}{\partial\tau} + \frac{T^3}{6}\frac{\partial \ddot{P}_0}{\partial\tau} + O(T^4)\right), \tag{21}$$

and it becomes clear that *the vortex sheet introduced upon the reduction of the slip velocity to zero after each projection cycle has the effect of adding—discontinuously, as a vortex sheet—another portion of the vorticity into* Ω *to that which was (smoothly) injected during the intermediate velocity phase of the projection cycle via the* $\tilde{\mathbf{u}} = \mathbf{w}$ *boundary condition.* For Projection 1 this is in fact *all* of the vorticity (none is injected during the first phase), via $s = -T\partial P_0/\partial\tau + O(T^2)$, whereas for higher-order projection methods it is a smaller correction term, since the major portion of vorticity input is accomplished via the better estimates of ∇P—and more smoothly at that. It is significant to note that the 'optimal' projection methods do no more than permit one-order (in T) higher vorticity input during the intermediate velocity phase by injecting more of the vorticity in a smoother way, resulting in a smaller-yet vortex sheet correction upon projection-plus-slip-velocity cancellation.

For the general case (curved boundaries in 3D) the same result holds. To see this, note first that the normal flux of tangential vorticity, $\mathbf{Q} \equiv -\nu\mathbf{n} \times \partial\omega/\partial n$, where $\omega \equiv \nabla \times \mathbf{u}$, becomes—using the identity $\partial\omega/\partial n = \nabla(\omega\cdot\mathbf{n}) + (\nabla \times \omega) \times \mathbf{n}$ and the fact that $\omega\cdot\mathbf{n} = 0$ on a stationary no-slip wall—$\mathbf{Q} = -\nu\mathbf{n} \times [(\nabla \times \omega) \times \mathbf{n}]$. But the viscous term in the NS equation is $\nu\nabla^2\mathbf{u} = -\nu\nabla \times \omega$ and it therefore follows that the tangential component of the momentum equations does indeed relate the tangential pressure gradient to the normal flux of tangential vorticity.

It therefore follows that the re-imposition of the no-slip BC for $\tilde{\mathbf{u}}$ at the beginning of each projection cycle is actually a *crucial* part of the projection method approximation and contributes vitally to its success. It may also be too harsh to label the projection slip velocity as spurious; it should be regarded as simply another aspect of the approximation—and that the comparison of $\mathbf{v}(T)$ with $\mathbf{u}(T)$ is simply 'awkward' as $x \to \Gamma$, where $\mathbf{u} \to \mathbf{w}$ smoothly but $\mathbf{v}$ suffers a jump (in the tangential direction)—and it is a larger jump using the simple BCs.

3.1.4. Outflow boundary conditions (OBCs). In order to generalize our results, we now drop the constraint $\Gamma_2 = \varnothing$ and show how the Neumann BCs (1d) are (or at least could be) accounted for in these projection methods; i.e. we return to the original problem given in (1). (We label the section OBCs because outflow situations account by far for the largest use of such BCs.) The incorporation of the Neumann BCs into the projection methods is based on (i.e. was derived from) an analysis of the weak form of the equations (as was (1d) derived, in point of fact) and is another example of the utility of weak forms; i.e. they can often show the way to useful and legitimate (natural) BCs. Only a summary of the results is given here; for a more detailed exposition, including the discrete case via the FEM, see Gresho *et al.*[19]

Before presenting the results—again in the form of algorithms—we remark that the most common/useful implementation of these Neumann BCs as OBCs is also the simplest: $F_n(t) = F_\tau(t) = 0$. $F_n = 0$ (or C, a constant) is appropriate for simple isothermal flows (no buoyancy effects) without body forces and only one outflow region (no splitting/branching of the domain), and tends to set the pressure level via $P \simeq -C$ and, concomitantly, yields $\partial u_n/\partial n \simeq 0$; i.e. it turns out that the viscous portion of the 'normal traction force' is usually small compared to the pressure part. $F_\tau = 0$ is useful in that it yields $\partial u_\tau/\partial n = 0$, and it is generally true that $\partial(\cdot)/\partial n = 0$ is a passive and useful OBC, at least when the outflow velocity is in (or close to) the same direction as the outward-pointing normal vector—even if only 'on average' as in, for example, vortex shedding.

The general projection methods that we advocate for problems with OBCs are as follows.

Projection 1 with OBC (22)

(0) Given $\mathbf{u}_0$ with $\nabla \cdot \mathbf{u}_0 = 0$,
(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu \nabla^2 \tilde{\mathbf{u}} = \tilde{\mathbf{f}} \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} \quad \text{on } \Gamma_1,$$

$$\nu \frac{\partial \tilde{u}_n}{\partial n} = F_n(t) \quad \text{and} \quad \nu \frac{\partial \tilde{u}_\tau}{\partial n} = F_\tau(t) \quad \text{on } \Gamma_2, \quad \text{for } 0 < t \leq T.$$

(2) Solve for φ from

$$\nabla^2 \varphi = \nabla \cdot \tilde{\mathbf{u}}(T) \quad \text{in } \Omega,$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \Gamma_1,$$

$$\varphi = -TF_n(T) \quad \text{on } \Gamma_2.$$

(3) Compute $\mathbf{v} = \tilde{\mathbf{u}}(T) - \nabla \varphi$ in $\bar{\Omega}$.
(4) Report $\mathbf{v}$; then set $t = 0$, $\mathbf{u}_0 = \mathbf{v}$ in Ω and on $\Gamma_2(\mathbf{u}_0 = \mathbf{w}(T)$ on $\Gamma_1)$ and go to step (1).

Remarks

1. If pressure is desired or required, it is given by $P(T) = \varphi/T$, a relationship that was already used to set φ on Γ_2 so that $P = -F_n$ there.
2. Same as Remarks 2–5 in the previous Projection 1 algorithm—see (18)—applied now on Γ_1.
3. Because we have done it, we can assert that the use of the BC $\varphi = 0$ on Γ_2 is actually also legitimate, even though it *always* implies $P = 0$ there—but it is legitimate *only* because it is saved by the biharmonic miracle, to be described later.

Projection 2 with OBC (23)

(0) Given $\mathbf{u}_0$ with $\nabla \cdot \mathbf{u}_0 = 0$ and P_0,
(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu \nabla^2 \tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \nabla P_0 \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} \quad \text{on } \Gamma_1,$$

$$\nu \frac{\partial \tilde{u}_n}{\partial n} = F_n(t) + P_0 \quad \text{and}$$

$$\nu \frac{\partial \tilde{u}_\tau}{\partial n} = F_\tau(t) \quad \text{on } \Gamma_2, \quad \text{for} \quad 0 < t \leq T.$$

(2) Solve for φ from

$$\nabla^2 \varphi = \nabla \cdot \tilde{\mathbf{u}} \quad \text{in } \Omega,$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \Gamma_1,$$

$$\varphi = -\frac{T}{2}[F_n(T) + P_0] \quad \text{on } \Gamma_2.$$

(3) Compute $\mathbf{v} = \tilde{\mathbf{u}}(T) - \nabla\varphi$ in $\bar{\Omega}$.
(4) Compute $P(T) = P_0 + 2\varphi/T$ in $\bar{\Omega}$.
(5) Report $\mathbf{v}$ and P; then set $t = 0$, $P_0 = P(T)$, $\mathbf{u}_0$ as in Projection 1 with OBC, and go to step (1).

Remarks

1. As in Remarks 1–5 of the previous Projection 2 algorithm—see (19).
2. As with Projection 1, the P–φ relationship of step (4) was used to set the φ BC on Γ_2, and they both derive from the desire to have $P = -F_n$ on Γ_2.
3. Same as Remark 3 for Projection 1 with OBC, except here $\varphi = 0$ implies $P(T) = P_0$.

Projection 3 with OBC (24)

(0) Given $\mathbf{u}_0$ with $\nabla \cdot \mathbf{u}_0 = 0$, P_0 and $\dot{P}_0$,
(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu\nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \nabla(P_0 + t\dot{P}_0) \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} \quad \text{on } \Gamma_1,$$

$$\nu\frac{\partial \tilde{u}_n}{\partial n} = F_n(t) + P_0 + t\dot{P}_0 \quad \text{and}$$

$$\nu\frac{\partial \tilde{u}_\tau}{\partial n} = F_\tau(t) \quad \text{on } \Gamma_2, \quad \text{for } 0 < t \le T.$$

(2) Solve for φ from

$$\nabla^2 \varphi = \nabla \cdot \tilde{\mathbf{u}}(T) \quad \text{in } \Omega,$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \Gamma_1,$$

$$\varphi = -\frac{T}{3}[F_n(T) + P_0 + T\dot{P}_0] \quad \text{on } \Gamma_2.$$

(3) Compute $\mathbf{v} = \tilde{\mathbf{u}}(T) - \nabla\varphi$ in $\bar{\Omega}$.
(4) Compute $P(T) = P_0 + T\dot{P}_0 + 3\varphi/T$ in $\bar{\Omega}$.
(5) Compute $\dot{P}(T) = [P(T) - P_0]/T$ in $\bar{\Omega}$.
(6) Report $\mathbf{v}$ and P; then set $t = 0$, $\mathbf{u}_0$ as in Projection 1 with OBC, $P_0 = P(T)$, $\dot{P}_0 = \dot{P}(T)$, and go to step (1).

Remarks. As in Remarks 1–7 of the previous Projection 3 algorithm with Γ replaced by Γ_1—see (20)—plus the second remark following the Projection 2 algorithm above.

Having now presented the complete algorithms proposed for 'solving' (1), we are (finally) ready to move on towards the 'explanations' alluded to/promised several times.

3.2. *Analysis of projection methods*

While the projection methods defined above require only periodic projections from $\tilde{\mathbf{u}}$ to $\mathbf{v}$ at the *discrete* times $t = nT$, $n = 1, 2, \ldots$, it is of course possible to *continuously* project $\tilde{\mathbf{u}}(t)$ to $\mathbf{v}(t)$ and thus to regard both $\mathbf{v}$ and φ as continuous functions of time. And this we now do (in principle) in order to analyse and better understand these schemes. Thus, focusing on Projection 2 (of the simpler algorithms) again, consider:

(0) Given $\mathbf{u}_0$ with $\nabla \cdot \mathbf{u}_0 = 0$ and P_0,

(1) Solve for $\tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ at $t = 0$, from

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu\nabla^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \nabla P_0 \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = \mathbf{w} \quad \text{on } \Gamma_1,$$

$$\nu\frac{\partial \tilde{u}_n}{\partial n} = F_n(t) + P_0 \quad \text{and} \quad \nu\frac{\partial \tilde{u}_\tau}{\partial n} = F_\tau(t) \quad \text{on } \Gamma_2.$$

(2) Perform the *continuous* projection; i.e.

(a) solve

$$\nabla^2\varphi = \nabla \cdot \tilde{\mathbf{u}}(t) \quad \text{in } \Omega,$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \Gamma_1,$$

$$\varphi = -\frac{t}{2}[F_n(t) + P_0] \quad \text{on } \Gamma_2;$$

(b) compute

$$\mathbf{v}(t) = \tilde{\mathbf{u}}(t) - \nabla\varphi(t) \quad \text{in } \bar{\Omega}.$$

(3) Compute

$$P(t) = P_0 + 2\varphi/t \quad \text{in } \bar{\Omega}.$$

(4) At $t = T$, set $\mathbf{u}_0 = \mathbf{v}(T)$ in Ω and on Γ_2, $\mathbf{u}_0 = \mathbf{w}(T)$ on Γ_1, $P_0 = P(T)$ in $\bar{\Omega}$, $t = 0$, and go to step (1).

The above theoretical/conceptual algorithm is presented solely for the purpose of the analysis that follows, but it is perhaps interesting to observe that it could also be used in practice.

A key feature of the analysis relates to the following question: 'Viewed as a continuous projection for $0 < t \leq T$, are there *additional* PDEs (besides those defined by the projection) that are satisfied by $\mathbf{v}$ and φ, and if so, what are they and what relationship—*if any*—do they bear to the NS equations?'

The first step towards the answer is to insert $\tilde{\mathbf{u}} = \mathbf{v} + \nabla\varphi$ into the PDE for the intermediate velocity to obtain

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)(\mathbf{v} + \nabla\varphi) = \tilde{\mathbf{f}} - \nabla P_0 \quad \text{in } \Omega, \tag{25a}$$

$$\mathbf{v}+\nabla\varphi = \mathbf{w} \quad \text{on } \Gamma_1, \tag{25b}$$

$$\nu\frac{\partial}{\partial n}[\mathbf{n}\cdot(\mathbf{v}+\nabla\varphi)] = F_n(t)+P_0 \quad \text{and} \quad \nu\frac{\partial}{\partial n}[\tau\cdot(\mathbf{v}+\nabla\varphi)] = F_\tau(t) \quad \text{on } \Gamma_2, \tag{25c}$$

which, when augmented by the initial condition $\mathbf{v}+\nabla\varphi = \mathbf{u}_0$ at $t = 0$, is a well-posed problem for the *linear combination* of the two vector fields $\mathbf{v}$ and $\nabla\varphi$, each component of which (e.g. $v_x+\partial\varphi/\partial x$) evolves independently if $\tilde{\mathbf{f}}$ is independent of $\tilde{\mathbf{u}}$; e.g. Stokes flow. Rearrangement yields

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla P_0+\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla\varphi = \nu\nabla^2\mathbf{v}+\tilde{\mathbf{f}}, \tag{26}$$

which we will refer to as a modified/*perturbed momentum equation* and note three things: (i) $\mathbf{v}$ could look like $\mathbf{u}$ to the extent that $P_0+(\partial/\partial t-\nu\nabla^2)\varphi$ looks like P; (ii) it could be solved for $\mathbf{v}(\mathbf{x}, t)$ if φ were known, using the (slippery but non-penetrating) BCs of (25b); which leads to (iii) another requirement then that $\mathbf{v}$ 'look like' $\mathbf{u}$ is, from (25b), that $\partial\varphi/\partial\tau$ is 'small' on Γ_1, and another, from (25c), is that both $\partial^2\varphi/\partial n^2$ and $\partial^2\varphi/\partial n\partial\tau$ are 'small' on Γ_2—all of which are most easily attained (in theory at least) by keeping T 'sufficiently small'.

Next we subtract the NS momentum equation (1a) from (26) to obtain

$$\frac{\partial(\mathbf{v}-\mathbf{u})}{\partial t}+\nabla\left[(P_0-P)+\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\varphi\right] = \nu\nabla^2(\mathbf{v}-\mathbf{u})+(\tilde{\mathbf{f}}-\mathbf{f}) \quad \text{in } \Omega, \tag{27}$$

which we analyse as follows.

1. Recall from (14b) that $\mathbf{v}-\mathbf{u} = O(t^3)$ outside the BL—where now this means $x > O[\sqrt{(\nu t)}]$, where x denotes the normal distance from Γ into Ω.
2. Using (again outside the BL) $\tilde{\mathbf{u}} = \mathbf{u}+(t^2/2)\nabla\dot{P}_0+O(t^3)$ in $\tilde{\mathbf{f}}-\mathbf{f} \equiv \mathbf{f}(\tilde{\mathbf{u}})-\mathbf{f}(\mathbf{u})$ leads to

$$\tilde{\mathbf{f}}-\mathbf{f} = \frac{t^2}{2}\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\cdot\nabla\dot{P}_0+O(t^3).$$

3. Finally, we have $P-P_0 = t\dot{P}_0+O(t^2)$.

Thus (27) can be approximated by

$$\nabla\left[\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\varphi-t\dot{P}_0\right] = O(t^2) \quad \text{in } \Omega, \tag{28}$$

at least outside the BL, which leads to the (important) result

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\varphi = t\dot{P}_0+O(t^2) \quad \text{in } \Omega, \tag{29a}$$

except perhaps within the BL.

This *parabolic* PDE for φ supplements the conventional (and elliptic) one $\nabla^2\varphi = \nabla\cdot\tilde{\mathbf{u}}$, and when provided with the BCs

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \Gamma_1 \tag{29b}$$

and

$$\varphi = -\frac{t}{2}[F_n(t)+P_0] \quad \text{on } \Gamma_2, \tag{29c}$$

and the initial condition

$$\varphi = 0 \quad \text{at } t = 0, \tag{29d}$$

would provide a well-posed problem for $\varphi(x, t)$ if indeed (29a) were valid in all of Ω.

Remark. Except for the BC on Γ_1, the 'optimal' scheme would imply a similar φ-problem; in fact, (29b) would then be replaced by $\partial\varphi/\partial n = (t^2/2)\partial\dot{P}_0/\partial n$, which is a *higher-order* term. Thus, to lowest order and 'away from Γ', φ is the *same* for the simpler scheme as for the optimal scheme.

So, to first order in t and outside the BL, φ is also governed by—or at least satisfies—the transient heat equation (29), whose solution in this case is closely approximated by the 'outer solution',

$$\varphi(\mathbf{x}, t) = \frac{t^2}{2}\dot{P}_0(\mathbf{x}) + O(t^3), \tag{30}$$

à la (14a), *provided* $x > O(\delta)$, where $\delta \equiv \surd(\nu t)$, which ensures that we are outside the BL. The solution for φ within the BL is not simple; it is also not (very) relevant as long as t is small enough—which we must and do assume.

This analysis *suggests* at least that non-optimal BCs for these projection methods can only be deleterious within a boundary layer of thickness $\surd(\nu t)$; outside this BL, φ returns to the same value—given by (30) and (14a)—that it would have using optimal BCs, which BCs are also not spared from *some* irregularity near Γ.

Remarks

1. The BL 'pollution' restricts *only* the normal component of $\nabla\varphi$ (from (30)) from applying inside the BL; the tangential component of $\nabla\varphi$ (again from (30)) applies all the way through the BL. Thus $\tau\cdot\mathbf{v} = \tau\cdot(\tilde{\mathbf{u}} - \nabla\varphi) = \tau\cdot\mathbf{w} - (t^2/2)\partial\dot{P}_0/\partial\tau + O(t^3)$ on Γ_1, *justifying our earlier assertions regarding the slip velocity* (e.g. Remark 4 of (19)).
2. If, however, the optimal BC had been used, (30) would apply without restriction throughout the BL—pollution then would only occur at higher order (in t) and the slip velocity would be smaller, $O(t^3)$, as shown earlier.
3. A similar analysis, with similar results, applies to the first- and third-order schemes; i.e. outside the BL, $(\partial/\partial t - \nu\nabla^2)\varphi = P_0 + O(t)$ and $\varphi(t) = tP_0 + O(t^2)$ for Projection 1, and $(\partial/\partial t - \nu\nabla^2)\varphi = (t^2/2)\ddot{P}_0 + O(t^3)$ and $\varphi(t) = (t^3/6)\ddot{P}_0 + O(t^4)$ for Projection 3. The φ–P relationships assumed in the recommended algorithms *have been justified*—at least 'away from' Γ.
4. The alleged 'bad' BCs implied for the pressure via Projections 1 and 2 are now vindicated by being interpreted in the following sense: while they are indeed bad and indeed used and enforced, they cause pollution *only* within the BL, outside of which the proper 'BCs' prevail—the quotation marks being meant to imply that we are aware of the slack use of the term *boundary* condition.

With this new information in hand we return to (26), which now reads

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla(P_0 + t\dot{P}_0) = \nu\nabla^2\mathbf{v} + \mathbf{f} + O(t^2), \tag{31}$$

and we see that $\mathbf{v}$ does indeed satisfy a PDE that is very close to the NS equation.

Finally, the pressure computation in the algorithm, $P(t) = P_0 + 2\varphi/t$, is seen to be justified by virtue of its consistency with (30) by using $\dot{P}_0 = [P(t) - P_0]/t + O(t)$. But the pressure actually *also* satisfies a higher-order PDE—because φ does—that will, in the semi-discrete case to follow, lead to a biharmonic equation; namely the divergence of (26), with $\nabla \cdot \mathbf{v} = 0$, gives

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\varphi = \nabla\cdot(\tilde{\mathbf{f}} - \nabla\dot{P}_0) \quad \text{in } \Omega. \tag{32}$$

This equation is endowed with the IC given by (29d) and the following BCs: (29b), (29c) *and*

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\frac{\partial\varphi}{\partial n} = \mathbf{n}\cdot\left(\nu\nabla^2\mathbf{v} + \tilde{\mathbf{f}} - \nabla P_0 - \frac{\partial \mathbf{v}}{\partial t}\right) \quad \text{on } \Gamma_1 \text{ and } \Gamma_2, \tag{33}$$

which, as is the case for the Neumann BC for the PPE (see GS), is a manifestation of $\nabla\cdot\mathbf{v} = 0$ on Γ that is obtained by applying the normal component of (26) on Γ. Outside the BL, (32) becomes $\nabla^2(P_0 + t\dot{P}_0) = \nabla\cdot\mathbf{f} + O(t^2)$ and (33) becomes $(\partial/\partial n)(P_0 + t\dot{P}_0) = \mathbf{n}\cdot(\nu\nabla^2\mathbf{v} + \mathbf{f} - \partial\mathbf{v}/\partial t) + O(t^2)$, which approximates (2), as desired/required.

When we return to the semi-discrete version of these higher-order equations we will see that φ satisfies a biharmonic equation that approximates (32), which will finally lead to the biharmonic miracle—the continuum version of which we have just endeavoured to uncover.

Thus, since this analysis and results generalize easily to the other projection schemes in this family, we have shown that *regardless of the BCs applied to* $\tilde{\mathbf{u}}$ *in the normal direction*, the projection methods described herein satisfy the relationships presented between φ and P and will therefore deliver *legitimate approximations* to the NS equations. While the normal pressure gradient (but *not* the tangential) will generally be perturbed by $O(1)$ on Γ, and higher-order normal derivatives may be very bad (e.g. $\partial^2 P/\partial n^2$ is $O(1/\delta)$ on Γ_1), the solution beyond δ (the 'outer solution') will be very good. The tangential BCs on $\tilde{\mathbf{u}}$ are *not* miraculously saved, however, and they need not be; here the use of physical BCs simply causes more of the wall vorticity generation to occur as vortex sheets. There appears, however, to be less freedom in the selection of $\tau\cdot\tilde{\mathbf{u}}$ than there is in that of $\mathbf{n}\cdot\tilde{\mathbf{u}}$. If optimal BCs were used for $\tilde{\mathbf{u}}$, the maximal compatibility condition would be satisfied and the boundary layer adjustment would be smaller—and so would the slip velocity.

It may be useful to repeat and emphasize at this point that none of the 'Projection 2' methods used to make computer codes (including our own) with which we are familiar[20-22] have used optimal BCs for $\tilde{\mathbf{u}}$. They all use $\tilde{\mathbf{u}} = \mathbf{w}$ and thus rely on the phenomena just presented to make their codes 'work'. Finally, although Chorin[2] and Kim and Moin[15] *did* attempt to use the optimal BCs for the simpler (one Poisson equation per cycle) Projection 1 method, we have shown already that the miracle is also required there since $\partial P/\partial n$ actually remains at its initial value during the entire flow evolution.

Remarks

1. We conjecture that these new results will help the mathematicians in their thus-far elusive proofs of convergence in the presence of boundaries. We hope that they will also be able to 'clean up' the somewhat heuristic analyses put forth here.
2. In Temam,[23] the 'bad' BC of Projection 1, $\partial P/\partial n = 0$, was also addressed, and it was shown, via 'higher-level' techniques using the tools of functional analysis applied to the semi-discrete equations, that 'this does not affect the convergence of the scheme'. We hope and believe that our approach complements his.
3. The effects of error accumulation should be, but have not yet been, accounted for; i.e. a *global* error analysis of these projection methods is desirable. (The analyses thus far

presented were only local in the sense that we assumed that the true NS velocity and pressure were available at the start of *each* projection cycle, when in fact they are not.) If the results of such (seemingly very difficult) efforts were to parallel those of the semi-discrete NS equations viewed as a differential–algebraic equation (DAE) system, they would predict error accumulation in velocity but not in pressure; this would then lower all $(\mathbf{v}-\mathbf{u})$ error estimates by one order in T.

3.3. Semi-discrete projection methods

While the 'proper' computer implementation of projection methods would *separate* projection error (the only error discussed thus far) from ODE integration error (the intermediate velocity must be time-integrated on the computer via an ODE method), no-one yet has been that smart. We too will simply (and stupidly (?), especially for flows that are approaching steady state and using Projection 2 or higher) assume, as (tacitly, usually) have all others, that the projection error is large enough that T should be no larger than Δt, the ('reasonable') time step of the selected ODE scheme, thus confounding the projection and ODE errors and leaving room for the development of still better methods, but generating simple algorithms that seem to work pretty well.

As is common practice, we will use semi-implicit techniques in which the viscous term is treated implicitly and the advection term explicitly. An important attribute of such semi-implicit methods is that they allow the solution of several *smaller*, *symmetric*, and *sequential* linear systems (one for $\tilde{u}$, one for $\tilde{v}$, one for φ) rather than a single, larger coupled system (e.g. for u, v and P). In fact, whether or not this was the principal objective of the early investigators of projection methods, *it is surely the only reason that we had for ever venturing into this deep intellectual morass*—we would *much prefer* to use 'honest GFEM', i.e. implicit methods on the fully coupled equations via the finite element method, and would always do so if our computer budget would allow it. We also prefer the trapezoid rule as the implicit technique, and use a modified forward Euler scheme for the explicit portion. The semi-discretized Projection 2 method that we have 'used' (i.e. for $\Delta x \to 0$) is the following.

Semi-discrete projection 2 with OBCs (34)

(0) Given $\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0 = 0$, solve for the initial pressure field from (2); i.e.

$$\nabla^2 P_0 = \nabla\cdot\mathbf{f}(\mathbf{u}_0) \quad \text{in } \Omega,$$

$$\frac{\partial P_0}{\partial n} = \mathbf{n}\cdot[\nu\nabla^2\mathbf{u}_0 + \mathbf{f}(\mathbf{u}_0) - \dot{\mathbf{w}}_0] \quad \text{on } \Gamma_1,$$

$$P_0 = \nu\frac{\partial u_{0n}}{\partial n} - F_n(0) \quad \text{on } \Gamma_2.$$

Set $m = 0$.

(1) Solve for $\tilde{\mathbf{u}}_{m+1}$ from

$$\frac{\tilde{\mathbf{u}}_{m+1} - \mathbf{u}_m}{\Delta t} + \nabla P_m = \tfrac{1}{2}\nu\nabla^2(\tilde{\mathbf{u}}_{m+1} + \mathbf{u}_m) + \mathbf{f}(\mathbf{u}_m) \quad \text{in } \Omega,$$

with

$$\tilde{\mathbf{u}}_{m+1} = \mathbf{w}_{m+1} \quad \text{on } \Gamma_1,$$

$$\nu\frac{\partial}{\partial n}(\mathbf{n}\cdot\tilde{\mathbf{u}}_{m+1}) = F_n(t_{m+1}) + P_m \quad \text{and}$$

$$\nu\frac{\partial}{\partial n}(\tau\cdot\tilde{\mathbf{u}}_{m+1})=F_\tau(t_{m+1})\quad\text{on }\Gamma_2;$$

i.e. solve

$$\left(I-\frac{\nu\Delta t}{2}\nabla^2\right)\tilde{\mathbf{u}}_{m+1}=\left(I+\frac{\nu\Delta t}{2}\nabla^2\right)\mathbf{u}_m+\Delta t(\mathbf{f}_m-\nabla P_m)$$

for $\mathbf{u}_{m+1}$ subject to the given BCs.

(2) Solve for φ from

$$\nabla^2\varphi=\nabla\cdot\tilde{\mathbf{u}}_{m+1}\quad\text{in }\Omega,$$

$$\frac{\partial\varphi}{\partial n}=0\quad\text{on }\Gamma_1,$$

$$\varphi=-\frac{\Delta t}{2}[F_n(t_{m+1})+P_m]\quad\text{on }\Gamma_2.$$

(3) Update the velocity from

$$\mathbf{u}_{m+1}=\tilde{\mathbf{u}}_{m+1}-\nabla\varphi\quad\text{in }\bar{\Omega}.$$

(4) Update the pressure from

$$P_{m+1}=P_m+2\varphi/\Delta t\quad\text{in }\bar{\Omega}.$$

Bump m and go to step (1).

Remarks

1. Again, $\partial P/\partial n|_{\Gamma_1}$ will remain unchanged from its initial value.
2. The analogous steps for the first- and third-order schemes should be obvious.
3. We have thus far only implemented and tested the first two of these projection schemes (Projections 1 and 2), although some years ago we designed a version of the Projection 3 scheme that used explicit time integration exclusively and did not use $T=\Delta t$. In Gresho *et al.*[16] we used an estimate of the local time truncation error to determine T, and forward Euler stability results to select Δt; the ratio $T/\Delta t$ was called the subcycle ratio, and ranged from 2 to 20 or more. (We also solved a pressure Poisson equation at each projection cycle rather than use the approximation $P=P_0+T\dot{P}_0+3\varphi/T$, which we did not then know.)
4. We actually overspecify the projected velocity in our codes; i.e. rather than *computing* $\tau\cdot\mathbf{u}_{m+1}=\tau\cdot(\tilde{\mathbf{u}}-\nabla\varphi)$ on Γ during the projection step, which would put the slip velocity in plain view, we hold $\tau\cdot\mathbf{u}_{m+1}=\tau\cdot\mathbf{w}_{m+1}$. This procedure is expedient and innocuous in the following sense: it is not hard to show that the error so incurred is at the level of the spatial truncation error multiplied by Δt; i.e. it vanishes with either temporal or spatial refinement. (It is *not* innocuous in the sense of spurious pressure modes when certain finite element methods are employed, a point we shall return to later.) This cost-effective short-cut is also justified *a posteriori* by the numerical results so obtained. In fact, as far as we can determine, only Zang and Hussaini[18] actually compute (and report?) the slip velocity. (Recall that it is the post-projection reduction to zero of the slip velocity that really matters.)
5. For reasons that we do not yet fully understand, the omission of the factor of 2 in step (4) seems to have little effect, i.e. $P_{m+1}=P_m+\varphi/\Delta t$ also works well in practice (finite Δx). We shall return to this point later—after presenting the fully discrete case in Part 2.

3.4. *A potentially more-cost-effective semi-discrete projection method*

In the light of previous discussion, it may be possible and useful to at least *attempt* to separate projection error from ODE error—especially, for example, when the Δt from the ODE is stability-limited and the flow approaches a steady state. We provide next some initial ideas in this direction, which we have not tested, but which with further development and testing might be cost-effective—a task we may leave to others. We utilize Projection 2 again. (Indeed, these ideas could probably not be fruitfully applied to Projection 1.)

For openers, suppose that Δt from the ODE scheme is so small that it may be assumed that the local truncation error is small enough to be 'negligible'—just the *opposite* of the assumption inherent in 'applied' projection methods. Also suppose that we monitor the size of $\nabla \cdot \tilde{\mathbf{u}}$ in some appropriate norm while integrating (at Δt) the intermediate velocity and project only when $\|\nabla \cdot \tilde{\mathbf{u}}\|$ reaches a maximum allowable value, $\|\nabla \cdot \tilde{\mathbf{u}}\|_{\max} \equiv \varepsilon$. (Wouldn't a 'colour movie' showing $\nabla \cdot \tilde{\mathbf{u}}$ in Ω be marvelous? And better yet, it could lead to better grid designs.) Recalling that $\tilde{\mathbf{u}} = \mathbf{u} + (t^2/2)\nabla \dot{P}_0 + O(t^3)$, we have $\|\nabla \cdot \tilde{\mathbf{u}}\| = O(t^2)$, and this leads easily to the following dynamic adjustment of the projection time T:

$$T_{k+1} = T_k \sqrt{[\varepsilon / \|\nabla \cdot \tilde{\mathbf{u}}(T_k)\|]}, \tag{35}$$

where T_0 (the value of T at the beginning of a simulation) is probably best determined as follows, recalling that the initial pressure field is already required as a part of the start-up procedure:

$$T_0 = \sqrt{[2\varepsilon / \|\nabla^2 \dot{P}_0\|)}, \tag{36}$$

where $\dot{P}_0$ is estimated as $(P_s - P_0)/\Delta t_s$ and where Δt_s is a (very) small time step used in a forward Euler scheme to go from P_0 to P_s; i.e. ($\Gamma_2 = \varnothing$ for simplicity):

(i) Compute

$$\mathbf{u}_s = \mathbf{u}_0 + \Delta t_s (\nu \nabla^2 \mathbf{u}_0 + \mathbf{f}_0 - \nabla P_0) \quad \text{in } \Omega,$$
$$\mathbf{u}_s = \mathbf{w}_s \quad \text{on } \Gamma.$$

(ii) Solve for P_s from

$$\nabla^2 P_s = \nabla \cdot \mathbf{f}_s \equiv \nabla \cdot \mathbf{f}(\Delta t_s) \quad \text{in } \Omega,$$
$$\partial P_s / \partial n = \mathbf{n} \cdot [\nu \nabla^2 \mathbf{u}_s + \mathbf{f}_s - (\mathbf{w}_s - \mathbf{w}_0)/\Delta t_s] \quad \text{on } \Gamma.$$

(iii) Compute $\nabla^2 \dot{P}_0$ from

$$\nabla^2 \dot{P}_0 = (\nabla^2 P_s - \nabla^2 P_0)/\Delta t_s = (\nabla \cdot \mathbf{f}_s - \nabla \cdot \mathbf{f}_0)/\Delta t_s.$$

Remarks

1. A useful value of ε would need to be determined, most likely by trial and error and hopefully not on a case-by-case basis. In general, of course, ε could be 'user-specified'.
2. If $T_{k+1} \gg \Delta t$, the assumptions made in estimating T_{k+1} are probably valid and the subcycling procedure described above would probably be effective; i.e. the next projection would then not be performed until $T_{k+1}/\Delta t$ steps had been taken in the $\tilde{\mathbf{u}}$-integraton.
3. If, on the other hand, $T_{k+1} \ll \Delta t$, and if the theory is still sound—probably a big 'if'—then Δt is too large (whether it be CFL-based or diffusion-based) and should probably be reduced to $O(T_{k+1})$. It may be the case that this situation would occur at initial start-up—especially if the transient implied by the BCs and ICs is 'sharp'.
4. The above 'first-cut' strategy ignored, among other things, the effect of local (and global)

ODE error in relating $\tilde{\mathbf{u}}$ to T; thus, for example, if $\Delta t \gg T_{k+1}$, the assumption upon which T_{k+1} was based $[\nabla \cdot \mathbf{u} \sim O(t^2)]$ is in error—perhaps fatally.
5. All things considered, probably the strategy should only be invoked when $T_{k+1} > O(\Delta t)$: it is then more justifiable and more cost-effective.
6. A similar device might be useful for Projection 3.

3.5. *The biharmonic miracle* (*BHM*)

An analysis of the semi-discrete Projection 2 method, analogous to that done earlier for the fully continuous case, is obviously of interest, since it is then but one 'small' step (spatial discretization) from there to an algorithm that can actually be programmed for the computer. It is also of interest because it reveals the BHM.

1. Insert $\tilde{\mathbf{u}}_{m+1} = \mathbf{u}_{m+1} + \nabla\varphi$ into the intermediate velocity equation of (34) to obtain the analogue of (25),

$$\left(I - \frac{\nu\Delta t}{2}\nabla^2\right)(\mathbf{u}_{m+1} + \nabla\varphi) = \left(I + \frac{\nu\Delta t}{2}\nabla^2\right)\mathbf{u}_m + \Delta t(\mathbf{f}_m - \nabla P_m),$$

which can be rearranged to

$$\frac{\mathbf{u}_{m+1} - \mathbf{u}_m}{\Delta t} + \nabla\left[P_m + \frac{1}{\Delta t}\left(I - \frac{\nu\Delta t}{2}\nabla^2\right)\nabla\varphi\right] = \tfrac{1}{2}\nu\nabla^2(\mathbf{u}_{m+1} + \mathbf{u}_m) + \mathbf{f}_m, \tag{37}$$

a semi-discrete modified/*perturbed momentum equation*—the analogue of (26).
2. Apply the divergence operator and use $\nabla \cdot \mathbf{u}_{m+1} = 0$ to get

$$(I - \delta^2\nabla^2)\nabla^2\varphi/\Delta t = \nabla \cdot (\mathbf{f}_m - \nabla P_m), \tag{38}$$

a biharmonic equation (finally!) for φ, where here the BL thickness is $\delta \equiv \sqrt{(\nu\Delta t/2)}$; cf. (32).
3. Use $\varphi = \Delta t(P_{m+1} - P_m)/2$ to obtain

$$\frac{\mathbf{u}_{m+1} - \mathbf{u}_m}{\Delta t} + \nabla\left(\frac{P_{m+1} + P_m}{2}\right) = \tfrac{1}{2}\nu\nabla^2(\mathbf{u}_{m+1} + \mathbf{u}_m) + \mathbf{f}_m + \delta^2\nabla\nabla^2\left(\frac{P_{m+1} - P_m}{2}\right)$$

from (37) and

$$\nabla^2\left(\frac{P_{m+1} + P_m}{2}\right) = \nabla \cdot \mathbf{f}_m + \delta^2\nabla^4\left(\frac{P_{m+1} - P_m}{2}\right)$$

from (38), which we 'interpret' as a momentum equation and a PPE respectively, each perturbed by a higher-order ($O(\Delta t^2)$) term. Further rearrangement leads to the biharmonic equation (BHE) that is actually satisfied by the *pressure*,

$$(I - \delta^2\nabla^2)\nabla^2 P_{m+1} = 2\nabla \cdot \mathbf{f}_m - (I + \delta^2\nabla^2)\nabla^2 P_m, \tag{39a}$$

which we also regard as a *perturbed PPE*.
4. The BCs associated with this BHE are

$$\frac{\partial P_{m+1}}{\partial n} = \frac{\partial P_m}{\partial n} = \left.\frac{\partial P}{\partial n}\right|_{t=0} \quad \text{on } \Gamma_1, \tag{39b}$$

$$P_{m+1} = -F_n(t_{m+1}) \quad \text{on } \Gamma_2, \tag{39c}$$

and $$\frac{\partial}{\partial n}(I-\delta^2\nabla^2)P_{m+1} = \mathbf{n}\cdot\left[\nu\nabla^2(\mathbf{u}_{m+1}+\mathbf{u}_m)+2\mathbf{f}_m-2\frac{\mathbf{u}_{m+1}-\mathbf{u}_m}{\Delta t}-(I+\delta^2\nabla^2)\nabla P_m\right] \quad \text{on } \Gamma_1 \text{ and } \Gamma_2, \tag{39d}$$

the last of which is the normal component of the modified momentum equation.

Remarks

1. As in the continuous case, the Neumann BC on Γ_1 is spurious.
2. Similar to the continuous case, the pressure field will 'adjust itself' from a spurious inner solution at and near the wall to a good outer solution outside the BL—an assertion we shall actually try to prove below.
3. φ from (38) 'looks like' φ from (32) after one time step starting from $\varphi = 0$.
4. After showing that the above BHE system is well-posed, we will introduce a 1D model that mimics this system and that we are able to solve—both analytically and numerically.

When $\Gamma_2 = \varnothing$—the only contentious case—a necessary condition that the BHE system have a solution is obtained by integrating (39a) over the domain to give, using the divergence theorem,

$$\int_\Gamma (I-\delta^2\nabla^2)\frac{\partial P_{m+1}}{\partial n} = 2\int_\Gamma \mathbf{n}\cdot\mathbf{f}_m - \int_\Gamma (I+\delta^2\nabla^2)\frac{\partial P_m}{\partial n}.$$

Invoking the third BC, (39d), then leads to the requirement that

$$\int_\Gamma \mathbf{n}\cdot[\nu\nabla^2(\mathbf{u}_{m+1}+\mathbf{u}_m)-2(\mathbf{w}_{m+1}-\mathbf{w}_m)/\Delta t] = 0.$$

But the viscous terms vanish via

$$\int_\Gamma \mathbf{n}\cdot\nabla^2\mathbf{u} = \int_\Omega \nabla\cdot(\nabla^2\mathbf{u}) = \int_\Omega \nabla\cdot[\nabla(\nabla\cdot\mathbf{u})-\nabla\times\nabla\times\mathbf{u}],$$

because $\nabla\cdot\mathbf{u} = 0$ and div curl $(\cdot) = 0$. The second term ($\dot{\mathbf{w}}$) vanishes by virtue of constraint (1h), and thus the biharmonic problem is well-posed.

We now turn to a 1D model problem that we believe sheds much light on the previous confusion. Consider the following two ODE problems.

(*1*) *The 'PPE'* (40)

$$-u'' = S = \text{constant} \quad \text{on } 0 < x < 1,$$
$$u' = a \quad \text{at} \quad x = 0,$$
$$u' = a - S \quad \text{at} \quad x = 1,$$

where we note that the solvability condition $\int_0^1 S = u'(0) - u'(1)$ is satisfied so that a solution exists—at least up to an arbitrary additive constant. This u, call it u_{PPE}, represents the good/desired solution.

(*2*) *The perturbed PPE or biharmonic equation* (*BHE*) (41)

$$-\left(I-\delta^2\frac{d^2}{dx^2}\right)u'' = \delta^2 u'''' - u'' = S \quad \text{on} \quad 0 < x < 1,$$
$$u' = q_0 \quad \text{at} \quad x = 0,$$

$$u' = q_1 \quad \text{at} \quad x = 1,$$

$$-\delta^2 u''' + u' = a \quad \text{at} \quad x = 0,$$

$$-\delta^2 u''' + u' = a - S \quad \text{at} \quad x = 1.$$

Remarks

1. We regard the BHE as 'spurious' and u from (41) as being in error to the extent that it differs from u_{PPE}.
2. The solvability condition $\delta^2 u'''|_0^1 - u'|_0^1 = S$ is satisfied and a solution exists—again up to an arbitrary additive constant.
3. Moreover, the solution exists *independently* of the values of q_0 and q_1.
4. It is noteworthy that the Neumann data of the unperturbed problem have appeared, in a perturbed form, in the higher-order BC, and that the true first-derivative BCs of the BHE are *unrelated* to those of the PPE.
5. Hopefully, it is clear that the dependent variable called u in the 1D problem corresponds to the pressure in the multidimensional problem.

The PPE solution is $u_{\text{PPE}} = ax - Sx^2/2$, and that of the BHE is

$$u = u_{\text{PPE}} + \frac{\delta}{e^{1/\delta} - e^{-1/\delta}} \{[S - a(1 - e^{-1/\delta}) + (q_1 - q_0 e^{-1/\delta})] e^{x/\delta}$$
$$+ [S + a(e^{1/\delta} - 1) + (q_1 - q_0 e^{1/\delta})] e^{-x/\delta}\},$$

which is easier to deal with in the (relevant) approximation that follows when we require $\delta \ll 1$:

$$u \simeq u_{\text{PPE}} + \delta[(S - a + q_1) e^{-(1-x)/\delta} + (a - q_0) e^{-x/\delta}], \tag{42}$$

a simple but (we claim) far-reaching result.

Remarks

1. The size of the perturbation (error) in u is $O(\delta)$ on Γ (i.e. at $x = 0$ or 1), that in u' (i.e. the normal pressure gradient) is $O(1)$, that in u'' (the Laplacian of P) is $O(1/\delta)$, etc.; i.e. progressively higher derivatives are progressively larger on Γ.
2. But there are thin BLs near both boundaries, and away from these, $u = u_{\text{PPE}} + O(\delta e^{-1/\delta})$; i.e. u is *very close* to u_{PPE} away from the 'walls'. Also away from the walls, $u' = u'_{\text{PPE}} + O(e^{-1/\delta})$, $u'' = u''_{\text{PPE}} + O(e^{-1/\delta}/\delta)$, etc.; the perturbations are noticeable only on and near Γ and drop to very small values in passing through the BL.
3. Whereas $u' = q_0$ at $x = 0$ and $u' = q_1$ at $x = 1$, as required by (41), it is also true that $u' \simeq a$ at $x = O(\delta)$ and $u' \simeq a - S$ at $x = O(1 - \delta)$; i.e. the *Neumann BC of the PPE* is nearly recovered upon passage through the BL—another result that is *independent* of the values of q_0 and q_1. The perturbed (third-derivative) BC associated with the perturbed PPE recovers the desired behaviour even though the actual Neumann BC satisfied by the BHE solution is no good.
4. $(I - \delta^2 d^2/dx^2)u = (I - \delta^2 d^2/dx^2)u_{\text{PPE}} = u_{\text{PPE}} + O(\delta^2)$, a result that will have direct application to the Projection 1 method.
5. If $q_0 = a$ and $q_1 = a - S$, the BHE actually has the *same* solution as the PPE, and the BHM is not required nor invoked. This is the analogue of the 'BC compatibility' discussed earlier—but it is not a perfect one since we can never obtain the *same* solutions in the multidimensional PDE case.

6. Finally, since $\delta^2 u'''' = O(e^{-1/\delta}/\delta)$ away from the walls, the BHE itself is effectively transformed to the PPE away from Γ.

This describes the BHM in 1D. To the extent that it carries over to the actual case of interest, the heretofore unexplained success of projection methods with non-optimal BCs is now explained.

In fact, the extension of the 1D analogy leads to another—though perhaps even less rigorous—BHM. Return to the modified momentum equation (37) and replace φ by $\Delta t(P_{m+1}-P_m)/2$ to obtain the *effective* pressure gradient $\nabla P_{\text{eff}} \equiv \nabla[P_m+(I-\delta^2\nabla^2)(P_{m+1}-P_m)/2]$. The extension of the 1D BHM—via (42) with $q_0 = q_1 = 0$—to the 2D (or 3D) pressure equation is then as follows.

(i) The approximate solution of the BHE—in the form presented in (38)—near Γ_1 is

$$\frac{P_{m+1}-P_m}{2} \simeq \left(\frac{P_{m+1}-P_m}{2}\right)_{\text{PPE}} + \delta e^{-x/\delta}\left[\frac{\partial}{\partial x}\left(\frac{P_{m+1}-P_m}{2}\right)_{\text{PPE}}\right]_{\Gamma_1},$$

where $[(P_{m+1}-P_m)/2]_{\text{PPE}}$ refers to the correct (NS) pressures and x is the normal distance from Γ_1 into Ω. (Note that we (still) have

$$\left.\frac{\partial}{\partial n}\frac{P_{m+1}-P_m}{2}\right|_{\Gamma_1} = 0,$$

the 'bad' Neumann BC.)

(ii) Insert this solution into $P_{\text{eff}} \equiv P_m+(I-\delta^2\nabla^2)(P_{m+1}-P_m)/2$ to get

$$P_{\text{eff}} \simeq P_m+(I-\delta^2\nabla^2)\left(\frac{P_{m+1}-P_m}{2}\right)_{\text{PPE}} - \delta^3 e^{-x/\delta}\frac{\partial}{\partial x}\left(\frac{P''_{m+1}-P''_m}{2}\right)_{\text{PPE},\Gamma_1},$$

where $P' \equiv \partial P/\partial \tau$ on Γ_1.

(iii) Next, since $[(P_{m+1}-P_m)/2]_{\text{PPE}}$ is (by assumption) *smooth*, it varies little in passing through the BL, which gives—*assuming* that $P_m \simeq P_m|_{\text{PPE}}$; i.e. we admit to applying simple local analysis when global analysis is actually called for—

$$P_{\text{eff}} \simeq \left(\frac{P_m+P_{m+1}}{2}\right)_{\text{PPE}} + O(\delta^2);$$

i.e. the *effective* pressure is a good approximation *even in the BL*, so that even $\partial P_{\text{eff}}/\partial n$ is accurate—another consequence/aspect of the BHM. (We note in passing that $P_{\text{eff}} \equiv (I-\delta^2\nabla^2)P_{m+1}$ for Projection 1, and the analogy with the 1D model problem (see Remark 4 after (42)) is especially clear since $(I-\delta^2\nabla^2)P_{m+1} = P_{\text{PPE}}+O(\delta^2)$, a result also established in Orszag *et al.*[4])

(iv) Finally, it follows—by inserting P_{eff} into (37)—that the 'perturbed' momentum equation mimics well the true momentum equation in *all* of Ω; only the tangential BC is different.

Before leaving the (continuous) 1D model, however, we should point out that we too worried about the important question: 'But suppose (after full discretization) that your mesh is too coarse to *resolve* the BL—i.e. suppose $\Delta x^2 \gg \nu\Delta t$, which could be a common occurrence in practice?' To answer this question, we used a finite difference method to generate approximate solutions of (40) and (41) (for $S = 6$, $a = 4$, $q_0 = q_1 = 0$) on grids containing 50, 100, 200 and 400 intervals and (for each grid, with $\Delta x = h$) δ/h-values of 0·01, 0·10, 1·0, 10 and 100 to cover the range from a *poorly resolved* BL (i.e. $\delta/h = 0{\cdot}01$ has $\sim$ zero BL resolution by the mesh) to a very-well-resolved one ($\delta/h = 100$ has $\sim$ 100 grid points in the BL), with the following results:

(i) In *all* cases the solution of the discrete BHE was close to that of the discrete PPE once out of the BL region.
(ii) For $\delta \ll h$ the solution of the discrete BHE ignores/does not recognize the existence of the BL that it cannot 'see'; i.e. the first node in from the boundary agrees well with the discrete solution of the PPE at the same node.

The good *and important* news from (ii) may even be termed the *discrete* BHM: *regardless* of the ratio of projection BL thickness δ to mesh spacing h, the projection method solution will mimic well the (discrete) NS solution for all node points farther than $O(\delta)$ from the boundary. It only remains a requirement to keep δ small.

To conclude the discussion of semi-discrete semi-implicit projection methods, it may be useful to restate them in such a way as to identify them with those previous researchers whose work we have studied. To this end we present the projection methods in the following 'combined' form ($\Gamma_2 = \varnothing$ here, partly for simplicity and partly because the treatment of Neumann BCs was not discussed in sufficient detail in the references).

1. Intermediate velocity:

$$\frac{\tilde{\mathbf{u}}_{m+1} - \mathbf{u}_m}{\Delta t} + \nabla(\alpha_1 P_m + \alpha_2 \Delta t \dot{P}_m) = \tilde{\mathbf{f}} + \nu \nabla^2 [\theta \tilde{\mathbf{u}}_{m+1} + (1-\theta)\mathbf{u}_m] \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}}_{m+1} = \mathbf{w}_{m+1} + \nabla\left(\beta_1 \Delta t P_m + \beta_2 \frac{\Delta t^2}{2} \dot{P}_m + \beta_3 \frac{\Delta t^3}{6} \ddot{P}_m\right) \quad \text{on } \Gamma.$$

2. Poisson equation:

$$\nabla^2 \varphi = \nabla \cdot \tilde{\mathbf{u}} \quad \text{in } \Omega,$$

$$\frac{\partial \varphi}{\partial n} = \frac{\partial}{\partial n}\left(\beta_1 \Delta t P_m + \beta_2 \frac{\Delta t^2}{2} \dot{P}_m + \beta_3 \frac{\Delta t^3}{6} \ddot{P}_m\right) \quad \text{on } \Gamma.$$

3. Final velocity:

$$\mathbf{u}_{m+1} = \tilde{\mathbf{u}}_{m+1} - \nabla \varphi \quad \text{in } \bar{\Omega}.$$

4. Update pressure terms and go to step 1. (Except as mentioned below, the pressure updates are as described earlier.)

Remarks

1. Advection is treated in various ways and is not included in our comparison.
2. $\theta = 0, \frac{1}{2}$ and 1 correspond to forward Euler, trapezoid rule and backward Euler respectively on the viscous term—and of course $\theta = 0$ is not a semi-implicit method.
3. The results are shown in Table I.
4. Orszag and co-workers[4,24,25] have used a related/similar projection method except that the 'fractional steps' are performed in a different order—typically (i) advection, (ii) projection, (iii) viscous. Also, 'intermediate' velocities are solved using the appropriate physical BCs and no pressure term (*à la* our simple Projection 1 method). See also Marcus,[26] in which a similar splitting method was found to be too inaccurate in some cases and was thus abandoned.
5. In some *explicit* projection methods (all of which are simply the forward Euler method)—e.g. Amsden and Harlow,[27] the 'two-step scheme' of Fortin *et al.*[17,28]—the paradox of $\partial P/\partial n = 0$ on Γ also arises (either explicitly by the author or, if not mentioned by the author, implicitly by the 'student'). We wish to point out *this* paradox is not at all related to a

Table I. Summary of semi-implicit projection methods

	Method	Parameters					
		α_1	α_2	β_1	β_2	β_3	θ
Optimal	P1	0	0	1	0	0	$\frac{1}{2}$
	P2	1	0	0	1	0	$\frac{1}{2}$
	P3	1	1	0	0	1	$\frac{1}{2}$
Investigator							
Chorin[2,3]	P1	0	0	1	0	0	1[a]
Fortin *et al.*[17]	P1	0	0	0/1[b]	0	0	1[a]
Temam[23]	P1	0	0	0	0	0	1
Temam[23,c]	P1	0	0	0/1[b]	0	0	1
Gresho *et al.*[16]	P3[d]	1	1	0	0	0	0
Kim and Moin[15]	P1	0	0	1	0	0	$\frac{1}{2}$
Van Kan[20]	P2	1	0	0	0	0	$\frac{1}{2}$
Zang and Hussaini[18]	P1	0	0	0/1[b]	0	0	$\frac{1}{2}$
Kim and Moin[21,e]	P2	1	0	0	0	0	$\frac{1}{2}$
Bell *et al.*[22,f]	P2	1	0	0	0	0	$\frac{1}{2}$
This work	P1[g]	0	0	0	0	0	$\frac{1}{2}$ or 1
	P2	1	0	0	0	0	$\frac{1}{2}$ or 1

[a] These authors used $\theta = 1$ (implicit Euler) in an ADI and/or fractional step context.
[b] $\beta_1 = 0$ in the normal direction, $\beta_1 = 1$ in the tangential.
[c] This family of schemes involves an arbitrary number of 'advection–diffusion' steps between t_m and t_{m+1} and could perhaps be referred to as Projection 1 with subcycling.
[d] This 'forward Euler + projection' scheme used subcycling in which 'many' smaller steps were used to advance $\tilde{\mathbf{u}}$. It is not a semi-implicit scheme and is included only because it is the closest approach to a 'Projection 3' scheme that we have seen. It differs too in that the pressure update was obtained by solving the PPE.
[e] This is their current method, except that their pressure update is unclear.
[f] Their pressure update is somewhat different: $P_{m+1/2} = P_{m-1/2} + \varphi/\Delta t$.
[g] To our knowledge, our simple treatment of Projection 1 ($\tilde{\mathbf{u}} = \mathbf{w}$ on Γ), implemented in the second part of this paper via an FEM, is the first that has been successfully demonstrated.

computational BL and the BHM; it has a completely different explanation, which is detailed in Gresho *et al.*[19] (See also Easton,[29] in which some 'MAC issues' were clarified, even though his terminology of 'homogeneous boundary conditions' is unconventional and confusing; 'a misnomer'—Easton.[30])

In the second part of this paper we shall demonstrate some of the techniques derived herein via a finite element method, although we hasten to point out that the theory derived in this paper is independent of the technique chosen for its implementation via spatial discretization.

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APPENDIX: SUMMARY OF PROPERTIES OF PROJECTION OPERATORS

General (continuum and discrete)	*Continuum*	*Discrete*
Projection to div-free subspace: $\wp^2 = \wp$, div $\wp = 0$	$\wp \equiv I - \nabla(\nabla^2)^{-1}\nabla\cdot$ curl $\wp$ = curl	$\wp \equiv I - M_L^{-1}CA^{-1}C^T$ —
$Q \equiv I - \wp$, $Q^2 = Q$; div Q = div	$Q \equiv \nabla(\nabla^2)^{-1}\nabla\cdot$	$Q \equiv M_L^{-1}CA^{-1}C^T$
$Q\wp = 0$: $w \equiv \wp v \rightarrow Qw = 0$; $\wp$ projects onto the null space of Q, i.e. onto the space of div-free vectors	$\nabla \cdot w = 0$	$C^T w = 0$
$\wp Q = 0$: $w \equiv Qv \rightarrow \wp w = 0$; Q projects onto the null space of $\wp$, i.e. onto the space of vectors that are gradients of scalars	$w = \nabla\varphi$ where $\nabla^2\varphi = \nabla\cdot v$ (also, curl $w = 0$)	$w = M_L^{-1}C\varphi$ where $A\varphi = C^T v$ —
If $u \equiv \wp v$ and $w \equiv Qv$, then $u + w = v$. Also, $u \perp w$, i.e. $\|u\|^2 + \|w\|^2 = \|v\|^2$, but only if $\mathbf{n}\cdot\mathbf{w} = 0$ on Γ (continuum)	$(a, b) \equiv \int_\Omega ab$, and $\|v\|^2 = (v, v)$	$(a, b) \equiv a^T M_L b$ and $\|v\|^2 = (v, v)$
The eigenvalues of $\wp$ are either 0 or 1 (ditto Q); $\|\wp\| = 1$ (ditto Q), so that the projections are norm-reducing: $w = \wp v \rightarrow \|w\| \leq \|v\|$ (ditto Q)		(A symmetric projection $\wp_2$ can also be constructed: $\wp_2 \equiv I - M_L^{-1/2}CA^{-1}C^T M_L^{-1/2}$, which gives orthogonality in L_2: $(a, b) \equiv a^T b$)

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